AN INTRODUCTION

TO

GEODETIC SURVEYING.

IN THREE PARTS:

I. THE FIGURE OF THE EARTH.

II. THE PRINCIPLES OF LEAST SQUARES.

III. THE FIELD WORK OF TRIANGULATION.

BY

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PREFACE.

IN the first part of this book several lectures on the Figure of the Earth, prepared as an introduction to a course of study in Geodesy, are republished. In the second part is given a condensed presentation of the fundamental principles and rules of the Method of Least Squares, written especially for students, surveyors, and engineers who are unable to spare the time required for the perusal of the larger books on the subject. The theoretical discussions have necessarily been omitted, but the fundamental ideas concerning adjustments, weights of observations, and probable error are fully explained, so that computations may be made intelligently and not blindly. In the third part is presented a synopsis of the methods and computations required in the field work of precise triangulation, particularly in secondary geodetic work. Care has been taken to illustrate the rules and formulas by numerical examples, and to give problems exemplifying their applications. Thus the three parts of the book form an introduction, both theoretical and practical, to the science of Geodesy, and especially to Geodetic Surveying.

MANSFIELD MERRIMAN.

August 5, 1892.

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THE FIGURE OF THE EARTH.

CHAPTER I.

THE EARTH AS A SPHERE.

1. WHEN surveying is carried on with such accuracy, or over so great areas, that it becomes necessary to take into account the curvature of the earth, it is called geodesy. The science of geodesy teaches how to conduct such measurements so that the relative positions of points, far removed it may be from each other on the earth's surface, can be accurately determined. For this purpose the figure of the earth must be known, at least approximately, and hence first in order in geodetic studies should be given some account of our present knowledge concerning its size and shape.

2. Were the surface of the earth a plane, as certain ancient peoples supposed, the science of geodesy could never have arisen, since measurements founded on the elementary geometry of Euclid would be capable of determining accurately its geographical features. In fact, however, such measurements become more or less entangled in discrepancies according to the size of the country over which they are carried. For instance, let three points be taken on the earth's surface at consider-

able distances apart; the sum of the three angles thus formed will be found, if measured by an instrument whose graduated arc is placed level at each station, to be greater than 180°. Or let us consider the system for the division of our public lands, the law concerning which provides that they shall be laid out into townships "six miles square," with sides running duly north and south or east and west. These two requirements, perfectly possible were the earth a plane, are in practice impossible, and the areas of the townships are only laid out "as nearly as may be" to the legally required quantity. From these and many other discrepancies we conclude that the earth's surface is not a plane.

3. Reasons for supposing the figure of the earth to be globular are given in all the text-books on astronomy.* They are: the appearance of the top of a light-house before its base to a ship approaching port, the dip of the sea horizon, the elevation of the pole star as we travel north, its depression as we travel back, and the new stars that come to view in the south, the analogy of the other planets, which, seen through a glass, seem to be globular, and, lastly, the circular form of the earth's shadow as observed in a lunar eclipse. To these must be added the well-known fact that travelers, going ever eastward, pass entirely round the earth, and return again to the point of starting. We regard it then as proved that the earth is globular; that is to say, like a globe; but whether spherical, or spheroidal, or ellipsoidal, or ovaloidal, there is thus far in our argument no evidence.

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* See NORTON'S *Astronomy* (Fifth Edition, New York, 1880), p. 2.

4. To obtain exact information regarding the figure of the earth, precise measurements on its surface are necessary. The most natural method of procedure is to assume the form to be spherical, and to test the hypothesis by observations; then, if this be found not satisfactory, to assume it spheroidal, and to make further measurements and calculations. This is the plan, in fact, which has been followed by scientists, and it is difficult indeed to conceive of one more feasible, since here, as in all science, each step in advance must be from the simpler to the more complex, and be suggested by the knowledge already attained. To assume the form spheroidal at first would be more or less impracticable too, for exact calculations regarding a spheroidal triangle, for instance, imply a knowledge of the eccentricity of the meridian ellipse, the very thing required to be found. In this chapter, then, we regard the earth as a sphere, and proceed to discuss the methods by which its size may be determined.

5. And first of all we must decide what is the surface whose form is to be investigated. This can be no other than that of the waters of the earth. The ocean covers fully three-fourths of the globe, its surface is regular compared to that of the land, and although it is agitated by winds and raised in tides, the position of its mean level is capable of being located very accurately. Moreover, the land is really elevated but little above the sea when compared with the great radius of the globe. The mean surface of the ocean is, then, the spherical surface whose radius is to be determined.

6. An approximate value for the radius of the globe may be found by observations made at sea upon the

distance of the visible horizon. It has been noted, for example, that two points distant about eight miles apart are just visible one to another when each is elevated ten feet above the sea level. Let a plane be passed through these two points and through the earth's center, and from the center let lines be drawn to the point of tangency and to one of the points of sight, forming a right-angled triangle, from which, with the given data, it is easy to find

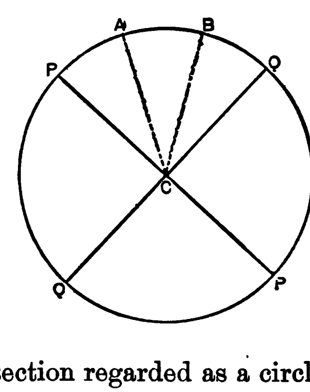
$$r = \text{about } 4200 \text{ miles}$$

for the radius of the globe. This value, as every one knows, is in excess by 200 miles or more, yet reflection upon the rude investigation leads us to two conclusions: first, that the earth is very large, and, secondly, that no precise estimation of its size can be deduced by observations of this kind. At an elevation of ten feet above the sea level vision is limited to a circle whose radius is about four miles, or whose area is about fifty square miles, while the whole surface is a million times as great. The highest mountains rise only about five miles, or about one eight-hundredth part of the radius. To conceive this slight elevation of the land imagine the earth to be reduced in size to a globe sixteen inches in diameter, then the tallest mountain would be only one one-hundredth of an inch in height—an amount scarcely perceptible to the eye. Since, then, the earth is so large, slight errors in the determination of the distance of the sea horizon are multiplied in the results, and such errors are particularly liable to occur, owing to the elevation of the visual line by the varying refraction of the atmosphere. The same objection may be made to methods founded on the measurement of the dip of the horizon, or on the vertical angles sometimes taken

in geodetic surveys for the determination of the relative heights of stations.

7. Regard now the earth from an astronomical point of view, as a globe revolving on an axis from west to east every twenty-four hours, and giving rise to an apparent rotation of the celestial sphere in the opposite direction. The invariable stars describe apparent circles around the celestial pole, and from the measured zenith distances of these stars as they cross the meridian the astronomical latitude of any place of observation may be found, by methods detailed in all the treatises on astronomy.* Let QPQP in the figure represent a section cut from the earth's sphere by a plane passing through the axis, that is the meridian section; PP representing the axis, QQ the equator, and C being the

Fig. 1.



center of the section regarded as a circle. Let A and B be two places on this meridian whose latitudes have been found (the angles ACQ and BCQ are these latitudes), then the angle ACB is known. Let also the

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* NORTON'S *Astronomy*, p. 72.

linear distance between A and B be measured. From these data the lengths of the whole quadrant and of the radius are easily found. Thus let φ be the angle ACB in degrees, and m the distance AB, then

$$\frac{m}{\varphi} = \text{length of one degree,}$$

$$90\,\frac{m}{\varphi} = \text{length of quadrant,}$$

$$57.2958\,\frac{m}{\varphi} = \text{length of radius,}$$

all being in the same unit of measure as the distance m.

To find, then, the size of the earth, measure the distance between two points on the same meridian, and find their difference of latitude. Such, in its simplest form, is the conception of the geodetic operation usually called the measurement of an arc of the meridian, the successful execution of which demands the most accurate instruments, the best observers, and long-continued labor. The determination of the difference of latitude is now usually made by zenith telescope observations at each station, and is perhaps the easiest part of the work. The length of the curved line of the meridian is more difficult to obtain, since it is usually impracticable to find a line of sufficient length running due north and south, and level enough to be directly measured with rods or chains. Ordinarily the two points are on different meridians, and the length of the meridian intercepted between their parallels of latitude is found by calculations from a triangulation carried on between them, the triangulation being itself calculated from the length of a measured base. But as a case where no triangulation is employed is the simpler, we choose such a one for the first illustration.

8. In the year 1763, the PENN family, proprietor of Pennsylvania and Delaware, and Lord BALTIMORE, proprietor of Maryland, employed two surveyors or astronomers, CHARLES MASON and JEREMIAH DIXON, to locate the boundary lines between their respective colonies. This work occupied several years, and while engaged upon it, MASON and DIXON noted that several of the lines, particularly the one between Maryland and Delaware, were well adapted to the determination of the length of a degree, being on low and level land, and deviating but little from the meridian. Representing this to the Royal Society of London, of which they were members, they received tools and money to carry on the work. The measured lines are shown in the annexed sketch. AB is the boundary between Delaware and Maryland, about 82 miles long and making an angle of about four degrees with the meridian; BD is a short line running nearly east and west; CD and PN are meridians about five and fifteen miles in length respectively; CP is an arc of the parallel, the same in fact as that of the southern boundary of Pennsylvania, the real "Mason and Dixon's line" of ancient American politics. In 1766 MASON and DIXON set up a portable astronomical instrument at A, the southwest corner of the present State of Delaware, and by observing equal altitudes of certain stars, determined the local time and the meridian, after which the azimuth of the line AB was measured, and the latitude of A found by observing the zenith distances of several stars as they crossed the meridian. At N, a point in the forks of the river Brandywine, the zenith distances of the same stars were also measured, from which it was easy to find the latitude of N, or the difference of latitude between A and N. In 1768 they made the linear measurements by means of wooden rectangu-

lar frames 20 feet in length. All the lines had in previous years been run in the operation for establishing the boundaries, and along each of them "a vista" cut, which * "was about eight or nine yards wide, and, in general, seen about two miles, beautifully terminating to the eye in a point." Toward this point they sighted the rectangular frames, brought one nicely into contact with the other, made them truly level, and noted the height of the thermometer in order to correct for changes due to expansion. Through the swamps they waded with the wooden frames, but across the rivers they found the distance by a simple triangle. Thus after many wearisome weeks and months the following values were deduced and sent home to England :

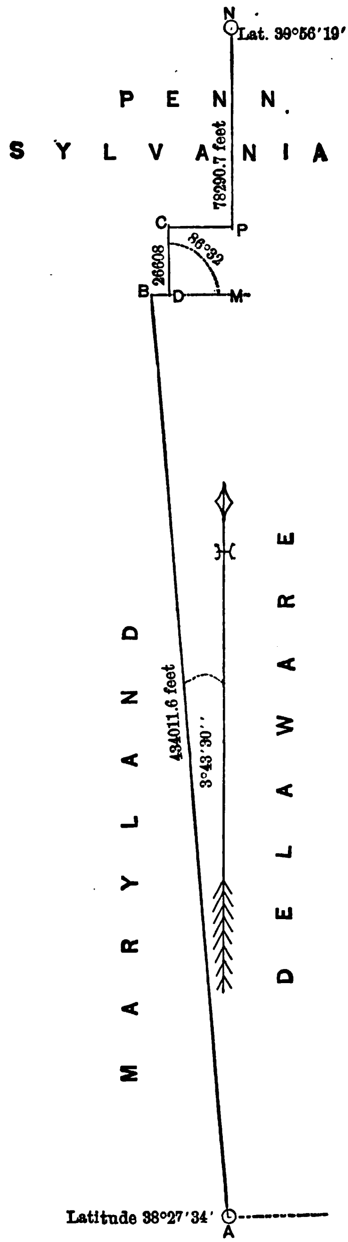
Latitude of	A = 38° 27′ 34″
Latitude of	N = 39 56 19
Diff. latitude	= 1 28 45

AB = 434011.6 English feet.
BD = 1489.9 " "
DC = 26608.0 " "
PN = 78290.7 " "
Azimuth of AB at A = 3° 43′ 30″ N. W.
Angle CDB = 93° 27′ 30″
CD and NP are true meridians.
CP an arc of parallel about 3 miles long.

Let us now find from these results φ the difference of latitude in degrees, and m the linear distance between the two stations A and N. The value of φ is

$$\varphi = 1^\circ.47917$$

* See London *Philosophical Transactions*, 1768, page 276. The data and results here given are taken from the articles of MASON and MASKELYNE in that volume.


N
Lat. 39°56′19″
P E N N.
S Y L V A N I A
78290.7 feet
C
P
88°32
26608
B
D
M
494011.6 feet
3°43′30″
M A R Y L A N D
D E L A W A R E
Latitude 38°27′34′
A

Now to find m, project, as in the sketch below, by arcs of parallels, each line upon a meridian passing through A. Then $m = AN'$, and this equals the sum of its parts $N'P'$, $P'D'$, $D'B'$, and $B'A'$, thus:

$$\begin{aligned} N'P' &= NP = 78290.7 \text{ feet.} \\ P'D' &= CD = 26608.0 \text{ “} \\ D'B' &= DG = \quad 89.8 \text{ “} \\ B'A' &= \quad 433078.8 \text{ “} \\ \hline m &= \quad 538067.3 \text{ feet.} \end{aligned}$$

(Here $D'B'$ or DG is found from the triangle BDG, taking it as plane, since its longest side is only 1490 feet long. But in finding AB' from the triangle BAB', where two of the sides are more than 80 miles long, AB and AB' are considered as arcs of great circles, and $B'B$ as an arc of a small circle of the sphere; to do this by the rules of spherical trigonometry involves a knowledge of the radius of the sphere, the very thing required to be found; but it is evident that only an approximate value is needed, and a few trials will show that the result for $B'A$ will come out the same within a small fraction of a foot, whether the radius of the earth be taken as 3800, 4000, or 4200 miles.) The length of one degree of the meridian now is

$$\frac{m}{\varphi} = 363764 \text{ feet} = 68.894 \text{ miles,}$$

from which we find the value

$$r = 3947.4 \text{ miles}$$

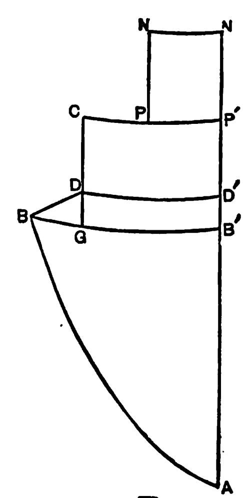


Fig. 3.

as the radius resulting from MASON and DIXON'S measurements. Since these were made on land elevated but slightly above the ocean, the result will not be materially lessened for a surface coinciding with the mean level of the waters of the earth.

9. But, as we know very well, a more accurate way of determining the distance between two distant points is by a triangulation. Here a long chain of triangles is formed, all the angles of which are carefully observed. One, at least, of the sides is located on a level plain, where it may be very precisely measured by special tools, and by finding the elevation above the ocean of the ends of this base, its length, and hence the whole triangulation may be reduced to that surface. Astronomical observations are made at several of the stations to determine their latitudes and the azimuth of the sides with reference to the meridian. The office work then begins. First, from the known lengths of the measured base and the known angles, the lengths of all the sides of the triangles and the positions of the stations are computed. A meridian is then conceived to be drawn north and south through the triangulation, as also parallels through each of the stations to meet this meridian, and the intercepted portions computed. The sum of these intercepts gives the length of the meridian between the northernmost and southernmost stations. Such operations, for instance, were carried on by French and Spanish scientists in Peru during the years 1736–40.* From Cotchesqui to Tarqui, a distance of about 220 miles, they set out stations forming forty-three triangles. Two of the sides of these triangles

* See *Histoire de l'Académie*, 1746, p. 618.

were carefully measured several times with wooden rods, the northern one near Cotchesqui being 6274.2 toises, and the southern one near Tarqui being 5259.95 toises. From these bases and the measured angles the length of the meridian between the two extreme stations was computed, and found to be

$$m = 176875 \text{ toises},$$

while from the astronomical observations the difference of latitude was

$$\varphi = 3^\circ\ 7'\ 3''.5 = 3^\circ.11764.$$

Hence the length of one degree of this arc is

$$56728 \text{ toises} = 68.702 \text{ miles},$$

and the corresponding value of the earth's radius is

$$3936.4 \text{ miles}.$$

The toise, we must here say, parenthetically, was an old French measure, now of classic interest on account of its use in this expedition and in the surveys made for deciding on the length of the meter; it is equal approximately to 1.949 meters, or 6.3946 English feet. The length of the degree and the radius resulting from the Peruvian arc, it must be mentioned, are not those of the ocean surface, since it lies on a high plateau, and the surveyors neglected to determine the elevation of their base lines.

10. It is now time that we should consider our subject more from a historical point of view, and attempt to give some account of the different efforts that have been made to determine the size of the earth. What

the Indian or Chinese nations have thought and done we know not; mainly from Europe come all the records, and in early times from Greece alone. ANAXIMANDER (year −570) speculated on the shape of the earth, and called it a cylinder whose height was three times its diameter, the land and sea being only upon its upper base, a view shared also by ANAXAGORAS (−460). PLATO (−400) thought it a cube. But ARISTOTLE (−340) gives good reasons for supposing it a sphere, and mentions, as also does ARCHIMEDES (−250), that geometers had estimated its circumference at 300000 stadia. ERATOSTHENES (−230) seems, however, to have been the first to conceive the principles and make the observations necessary for a logical deduction of the size of the sphere. He noticed that at Syene, in Southern Egypt, the sun at the summer solstice cast no shadow of a vertical object, it being directly in the zenith, while at Alexandria, in Northern Egypt, the rays of the sun at the same time of the year made an angle with the vertical of one-fiftieth of four right angles. From this he concluded that the circumference of the earth was fifty times the distance between these two places, and this being, according to the statements of travelers, 5000 stadia, he claimed for the whole circumference 250000 stadia. The exact length of the stadia is now unknown, so that we cannot judge of the accuracy of his result; it is probably much too large, since PTOLEMY, a learned astronomical writer, who flourished four hundred years later, mentions 180000 stadia as the length of the circumference; yet the name of ERATOSTHENES will ever be honored in science as that of the originator of the method of deducing the size of the earth from a measured meridian arc. POSIDONIAS (−90) made also similar observations between Alexandria and Rhodes, using a

star, instead of the sun, to find the difference of latitude, and deduced 240000 stadia for the circumference. But this knowledge of the Greeks was all lost as their civilization declined, and for more than a thousand years Europe, sunk in intellectual darkness, made no inquiry concerning the size or shape of the earth. Only in Arabia were the sciences at all cultivated during this period. There the Caliph ALMAMOUN summoned to Bagdad astronomers, and one of their labors was the measurement, on the plains of Mesopotamia, of an arc of a meridian by wooden rods, from which they deduced the length of a degree to be 56⅔ Arabian miles—probably about 71 of our miles.

11. In the fifteenth century, when the first gleams of light broke in upon the darkness of the middle ages, men began to think again about the shape of the earth. Navigators began to doubt that its surface was a level plane, and here and there one, like COLUMBUS, asserted it to be globular. In the sixteenth century, the learned accepted again the doctrine of the spherical form of the earth, and one of the ships of MAGELLAN, after a three years' voyage, accomplished its circumnavigation. With the acceptance of this idea arose also the question as to the size of the globe, and FERNEL, in 1525, made a measurement of an arc of a meridian by rolling a wheel from Paris to Amiens to find the distance, and observing the difference of latitude with large wooden triangles, from which he deduced about 57050 toises for the length of one degree. At this time methods of precision in surveying were entirely unknown. In 1617 SNELLIUS conceived the idea of triangulating from a known base line, and thus, near Leyden, he measured a meridian arc which gives 55020 toises for the length of a degree.

NORWOOD, in 1633, chained the distance from London to York, and deduced 57424 toises for a degree. PICARD, who was the first to use spider lines in a telescope, remeasured, in 1669, the arc from Paris to Amiens, using a base line and triangulation, and found one degree to be 57060 toises. This was the result that NEWTON used when making his famous calculation which proved that the moon gravitated toward the earth. In 1690–1718 CASSINI carried on surveys in France, more accurate, probably, than any of the preceding ones, and in 1720 he published the following results:

Arc.	Mean Latitude.	Length of 1°.
1.	49° 56′	56970 toises
2.	49° 22′	57060 "
3.	47° 57′	57098 "

and from these it appeared that the length of a degree of latitude increased toward the equator and decreased toward the poles, or, in other words, that the earth was not spherical, but spheroidal, and that the spheroid was prolate or extended at the poles. From the time men had ceased to believe in the flatness of the earth, and had begun to regard it as a sphere, their investigations had been directed toward its size alone; now, however, the inquiry assumed a new phase, and its shape came up again for discussion.

12. We must here interrupt the historical narrative to say a word about spheroids. A prolate spheroid is generated by an ellipse revolving about its major axis, and an oblate spheroid by an ellipse revolving about its

minor axis. The upper diagram in Fig. 4 represents a meridian section of the earth regarded as a prolate, and the lower shows it as an oblate spheroid. In each diagram PP is the axis, QQ the equator, C the center, A a

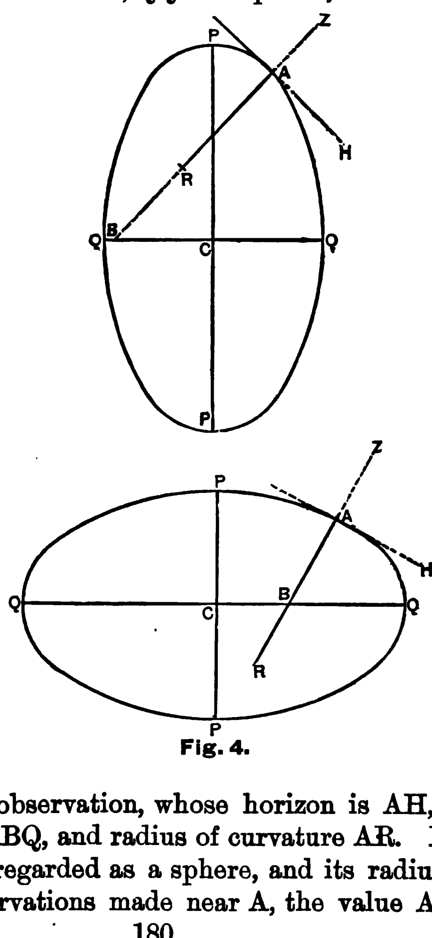


Fig. 4.

place of observation, whose horizon is AH, zenith Z, latitude ABQ, and radius of curvature AR. Now, if the earth be regarded as a sphere, and its radius be found from observations made near A, the value AR will result, it being always $\frac{180}{\pi}$ times the length of one degree

of latitude at A. In the prolate spheroid the radius of curvature is least at the poles and greatest at the equator, and the reverse in the oblate. Hence if the lengths of the degrees of latitude decrease from the equator to the poles, it shows that the earth is prolate; but if they increase from the equator toward the poles, it is a proof that it is oblate in shape.

13. Let us now go back to the year 1687, the date of the publication of the first edition of NEWTON'S *Principia*. In Book III. of that great work are discussed the observations of RICHER, who, having been sent to Cayenne, in equatorial South America, on an astronomical expedition, noted that his clock, which kept accurate time in Paris, there continually lost two seconds daily, and could only be corrected by shortening the pendulum. Now, the time of oscillation of a pendulum of constant length depends upon the intensity of the force of gravity, and NEWTON showed, after making due allowance for the effect of centrifugal force, that the force of gravity at Cayenne, compared with that at Paris, was too small for the hypothesis of a spherical globe; in short, that Cayenne was further from the center of the globe than Paris, or that the earth was an oblate spheroid, flattened at the poles. He computed, too, that the amount of this flattening at both poles was between $\frac{1}{180}$ and $\frac{1}{300}$ of the whole diameter. Now it will be remembered that NEWTON'S philosophy did not gain ready acceptance in France; this investigation, in particular, called forth much argument, and when CASSINI'S surveys were completed, indicating a prolate spheroid, the discussion became a controversy. Then the French Academy resolved to send out two expeditions to make measurements of meridian arcs that would definitely settle the matter,

one to the equator and another as far north as possible; for it was evident that observations near the latitude of France could afford only unsatisfactory information concerning the ellipticity of the meridian. Accordingly two parties sailed in 1735—MAUPERTIUS to Lapland, and BOUGUER and LACONDAMINE to Peru. MAUPERTIUS measured his base upon the frozen surface of the river Tornea, executed his triangulation and latitude observations, and returned to France in less than two years. The Peruvian expedition, whose work we have already described, was absent about seven years, but upon its return the following results could be written:

Arc.	Mean Lat.	Length of 1° of Latitude.
Lapland	N. 66° 20′	57 438 toises
France	N. 49° 22′	57 060 "
Peru	S. 1° 34′	56 728 "

These figures decided the question; from that time on, every one has granted that the earth is an oblate spheroid, rather than a sphere or a prolate spheroid.

14. Our consideration of the earth as a sphere is not yet finished, and it cannot be here completed without anticipating to a certain extent some of the results of the following chapters. What has already been said is sufficient for us to observe that the amount of flattening at the poles, and the deviation from the spherical form is not large. In fact, on a globe sixteen inches in equatorial diameter, and on which the thickness of a coat of varnish would represent the elevation of the lands above the waters, the polar axis would be 15.945 inches, or, in other words, the difference between the polar and

equatorial diameters would be but one-eighteenth of an inch. It is hence evident that for many purposes it is sufficiently accurate to consider the earth as a sphere. What value, then, shall we take for its radius, and what is the mean length of a degree of latitude on its surface?

15. The mean length of a degree of latitude is the average of the lengths of all the degrees from the equator to the poles, or one-ninetieth of the elliptical quadrant. Now the following are some of the values of the length of the elliptical quadrant, according to the calculations of mathematicians, made by methods which will be explained in the next chapter:

Year.	By whom.*	Quadrant in Meters.
1806	DELAMBRE	10 000 000
1819	WALBECK	10 000 268
1830	SCHMIDT	10 000 075
1831	AIRY	10 000 976
1841	BESSEL	10 000 856
1856	CLARKE	10 001 515
1866	CLARKE	10 001 887
1868	FISCHER	10 001 714
1872	LISTING	10 000 218
1878	JORDAN	10 000 681
1880	CLARKE	10 001 868

It will be seen from this table that scientists are by no means yet able to agree upon the length of the quadrant to single meters, or tens, or hundreds of meters. We select the value of BESSEL, 10 000 856 meters, for two reasons, first and mainly, because this and the other

* See the next chapter for references to some of the original memoirs and discussions of these investigators.

dimensions of the spheroid as deduced by BESSEL have been long in use in geodetic computations, and are now still very much used, notwithstanding all the later investigations; and, secondly, because in regarding the earth as a sphere, it makes little difference in our results whichever value be taken (and, curiously, the average of the above eleven values is 10 000 914 meters, or nearer to BESSEL's value than to any other). The mean length of one degree is, then,

$$\frac{10\ 000\ 856}{90} = 111\ 121 \text{ meters.}$$

From this is deduced the following useful table of mean length of arcs of latitude:

Length of	In Meters.	In Feet.
One degree	111 121	364 574
One minute	1 852	6 076
One second	30.9	101.3

The mean length of one degree in statute miles is 69.043 or $69\frac{1}{22}$. As the probable error of BESSEL's value of the quadrant is about 500 meters, the probable error of the above mean length of one degree is about 5.5 meters or 18 feet.* Stated in round numbers, easy to remember, the result is:

$$1^\circ \text{ of latitude} = 111.1 \text{ kilometers} = 69 \text{ miles.}$$

16. The mean radius of the earth, considered as a sphere, can be nothing more than the arithmetical mean

* For definition of probable error, and methods of determining it, see MERRIMAN's *Elements of the Method of Least Squares* (New York, 1877), pp. 19, 58, and 94.

or average of all the radii of the spheroid. A moment's reflection will convince us that this mean radius is the same as the radius of a sphere having a volume equal to the volume of the spheroid. Let a be the equatorial and b be the polar radius of the oblate spheroid, equal, according to BESSEL, to 6 377 397 and 6 356 079 meters respectively; the volume is $\frac{4}{3}\pi a^2 b$. Let r be the radius of the sphere whose volume is $\frac{4}{3}\pi r^3$. Place these values equal, and we have

$$r = \sqrt[3]{a^2 b},$$

which gives

$$r = 6\ 370\ 283 \text{ meters},$$

or in round numbers,

$$r = 6\ 370 \text{ kilometers},$$
$$r = 20\ 899 \text{ thousand feet},$$
$$r = 3\ 958 \text{ statute miles}$$

for the mean radius of the waters of the earth.

17. This mean value of r is, however, incongruous with the above mean length of a degree of latitude, for the quadrant of a circle corresponding to a radius of 6 370 kilometers is nearly 6 kilometers greater than BESSEL's elliptical quadrant of 10 000 856 meters. In some kinds of map projections it may be more logical to use the radius of a circle whose circumference is equal to the circumference of the meridian ellipse; this requires the equation

$$\tfrac{1}{2}\pi r = 10\ 000\ 856 \text{ meters},$$

from which

$$r = 6\ 366\ 743 \text{ meters},$$

or in round numbers,

$$r = 6\ 367 \text{ kilometers} = 3\ 956 \text{ miles},$$

which is less by two miles than the mean radius of the sphere. This discrepancy is unavoidable, since the properties of a sphere and an ellipsoid are not the same. At the beginning of our discussion we saw that the earth's surface could not be plane because of the discrepancies of surveys with the geometry of the plane, and here we see that it is also impossible, when precision is demanded, to consider it as spherical. Therefore, whenever in any problem a variation of two or three miles in the length of the mean radius would make any practical change in the result of the solution, it is better to regard the earth as an oblate spheroid, and this we shall discuss in the next chapter.

CHAPTER II.

THE EARTH AS A SPHEROID.

18. SINCE an oblate spheroid is a solid generated by the revolution of an ellipse about its minor axis, the equator and all the sections of the spheroid parallel to the equator are circles; and all sections made by planes passing through the axis of revolution are equal ellipses. Let a and b represent the lengths of the semi-major and semi-minor axes of this meridian ellipse, which of course are the same as the semi-equatorial and semi-polar diameters of the spheroid; when the values of a and b have been found all the other dimensions of the ellipse and the spheroid become known. At first we must express algebraically the properties of the ellipse; then combining some of these with the data deduced by measurements we find, as was done in the last chapter for the circle, the form and size of the earth's meridian section.

19. The eccentricity and ellipticity of an ellipse are merely two fractions, the first defined by the equation

$$e = \frac{\sqrt{a^2 - b^2}}{a},$$

and the second by

$$f = \frac{a - b}{a};$$

or, in other words, the eccentricity e is the distance from the center of the ellipse to one of the foci divided by the semi-major axis, and the ellipticity f is the amount of flattening at one of the poles divided by the semi-major axis. The relation between these two fractions is easy to deduce, namely:

$$f = 1 - \sqrt{1 - e^2},$$

or

$$e = \sqrt{2f - f^2}.$$

From the definitions of e and f we may express b in terms of a as follows:

$$b = a\sqrt{1 - e^2},$$

or

$$b = a(1 - f).$$

The two quantities relating to the ellipse that we shall need most particularly to use are the length of the quadrant and of the radius of curvature at any point. These are deduced in many text-books* by means of the calculus; we here simply note their values and consider them as proved. The length of the quadrant is

$$q = \frac{a\pi}{2}\left(1 - \frac{e^2}{4} - \frac{3e^4}{64} - \ldots\right),$$

or perhaps more conveniently

$$q = \frac{a\pi}{2}\left(1 - \frac{f}{2} + \frac{f^2}{16} - \ldots\right).$$

* See, for instance, CLARK'S *Calculus* (Cincinnati, 1875), pp. 190, 342.

If l be the latitude of any point on the meridian ellipse, the radius of curvature of the curve at that point is

$$r = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2 l)^3}}.$$

For the equator, where $l = 0°$, this has its least value $\frac{b^2}{a}$; but for the poles, where $l = 90°$, it has its greatest value $\frac{a^2}{b}$. Now in determining the form and size of the ellipse we may seek a and b, or any two convenient functions of a and b. Those usually employed are a and e; when these have been found, b and q and f and r are also known from the above equations.

20. Were the earth a perfect sphere, one arc of a meridian measured with precision would be enough to deduce the value of its radius. As it is, however, plainly a spheroid, and as a spheroid requires two dimensions for establishing its size, it would seem that two measured arcs of meridians are at least required. Let m_1 and m_2 be the measured lengths of two meridian arcs, φ_1 and φ_2 their amplitudes, that is, the number of degrees of latitude between their northern and southern extremities, l_1 and l_2 their middle latitudes, r_1 and r_2 the radii of curvature of their middle points. Regarding these arcs as arcs of circles, their radii of curvature are

$$r_1 = \frac{180}{\pi} \cdot \frac{m_1}{\varphi_1},$$

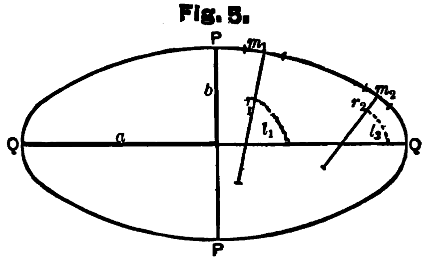
$$r_2 = \frac{180}{\pi} \cdot \frac{m_2}{\varphi_2}.$$

Considering now the middle points of these arcs as lying upon the circumference of an ellipse whose semi-major axis is a, and eccentricity e, these radii are

$$r_1 = \frac{a\,(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 l_1)^3}},$$

$$r_2 = \frac{a\,(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 l_2)^3}}.$$

Fig. 5.



By equating the two values of r_1, and also the two values of r_2, we have the following conditions:

$$\frac{180}{\pi}\,\frac{m_1}{\varphi_1} = \frac{a\,(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 l_1)^3}},$$

$$\frac{180}{\pi}\,\frac{m_2}{\varphi_2} = \frac{a\,(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 l_2)^3}},$$

which contain eight quantities, all known except a and e. It is evident that a and e will be the more accurately determined the nearer to the pole one of the arcs be taken, and the nearer to the equator the other. To solve these equations, observe that if the first be divided by the second, we obtain an equation containing e^2 alone, from which

$$e^2 = \frac{1 - \left(\frac{m_1\varphi_2}{m_2\varphi_1}\right)^{\frac{2}{3}}}{\sin^2 l_2 - \left(\frac{m_1\varphi_2}{m_2\varphi_1}\right)^{\frac{2}{3}}\sin^2 l_1}.$$

Then to find a, place the value of e in either of the above equations and solve for a.

21. For an example let us take the two arcs measured about the year 1737, by astronomers in the employ of the French Academy, one in Lapland, and the other in Peru. The data are as follows:

Lapland Arc:

Length = 92 778 toises = 180 827.7 meters.
Lat. of N. end = + 67° 8′ 49″.83.
Lat. of S. end = + 65° 31′ 30″.26.

Peruvian Arc:

Length = 176 875.5 toises = 344 736.8 meters.
Lat. of N. end = + 0° 2′ 31″.39.
Lat. of S. end = − 3° 4′ 32″.07.

Calling the Lapland Arc No. 1, and the Peruvian No. 2, we find l_1 and l_2 by taking the mean of the two latitudes in each case, and φ_1 and φ_2 by taking their difference. Then

$$\begin{aligned}
m_1 &= 180\ 827.7 \text{ meters}, \\
\varphi_1 &= 1^\circ.6221, \\
l_1 &= +\ 66^\circ\ 20'\ 10''.05, \\
m_2 &= 344\ 736.8 \text{ meters}, \\
\varphi_2 &= 3^\circ.1176, \\
l_2 &= -\ 1^\circ\ 31'\ 0''.34.
\end{aligned}$$

Substituting these values in the above expression for e^2 we find

$$e^2 = 0.00643506;$$

or $$e \doteq 0.08022.$$

Inserting this value of e^2 in either of the original equations, and solving for a, we find

$$a = 6\,376\,568 \text{ meters.}$$

From the value of e^2 we find also

$$f = 0.0032228,$$

and then $$b = 6\,356\,020 \text{ meters},$$
$$q = 10\,000\,150 \text{ meters},$$

and these values fully determine the oblate spheroid corresponding to the two meridian arcs. It is often customary to state the value of the ellipticity as a vulgar fraction whose numerator is unity, since thus a clearer idea is presented of the flattening at the poles. In this case the decimal fraction 0.003223 gives

$$f = \frac{1}{310.3};$$

that is, the amount of the flattening at one of the poles is about $\frac{1}{310}$th of the equatorial radius. In the same way the eccentricity may be written

$$e = \frac{1}{12.5};$$

or the distance of the focus of the ellipse from the center is about $\frac{1}{12.5}$th of the equatorial radius. These fractions are both somewhat too small for the actual spheroid, as will be shown in future paragraphs.

22. Let us now go back to the year 1745, or there-

abouts, when, it will be remembered, the results of the surveys instituted by the French Academy became known. These results have been stated in the previous chapter in toises; we note them here again in meters:

Arc.	Mean Latitude.	Length of 1° of Latitude.
		Meters.
Lapland	+ 66° 20′	111 949
France	+ 49° 22′	111 212
Peru	− 1° 34′	110 565

By the method above explained, or by other similar methods, these data may be combined in three different ways to deduce the shape and size of the earth, assuming it to be a spheroid of revolution. These combinations gave for the ellipticity values about as follows:

from Lapland and French Arcs, $\frac{1}{146}$,
from Lapland and Peruvian Arcs, $\frac{1}{310}$,
from French and Peruvian Arcs, $\frac{1}{334}$.

Now if the earth be a spheroid of revolution, and if the measurements be well and truly made, then these values of the ellipticity should be the same. As, however, they disagree, the conclusion is easy to make that either the assumption of a spheroidal surface is incorrect, or the surveys are inaccurate. To settle this question there were measured in the following fifty years a number of meridian arcs in different parts of the world, one in South Africa by Lacaille, one in Italy by Boscovich, one in America by Mason and Dixon, one in Hungary by Liesganig, and one in Lapland by Svanberg, while in France, England, and India, geodetic surveys furnished also the materials for the deduction of other arcs. Most

important of all was the investigation undertaken by the French for the derivation of the length of the meter, the surveys for which, with the accompanying office-work lasted from 1792 to 1807. This work was under the charge of the celebrated astronomers DELAMBRE and MÉCHAIN, and the meridian arc extended from the latitude of Dunkirk on the north, to that of Barcelona on the south, embracing an amplitude of nearly ten degrees. In this survey the methods for the measurement of bases and angles were greatly improved, and, in fact, here approached for the first time to modern precision. The results, as finally published in 1810, were,

$$\begin{aligned}\text{length of arc} &= 551\,584.7 \text{ toises},\\ \text{amplitude} &= 9^\circ\ 40'\ 23''.89,\end{aligned}$$

and these were combined with the corresponding values in the Peruvian arc to find the ellipticity. The combination gave

$$f = \frac{1}{334},$$

and then the length of the quadrant was found to be

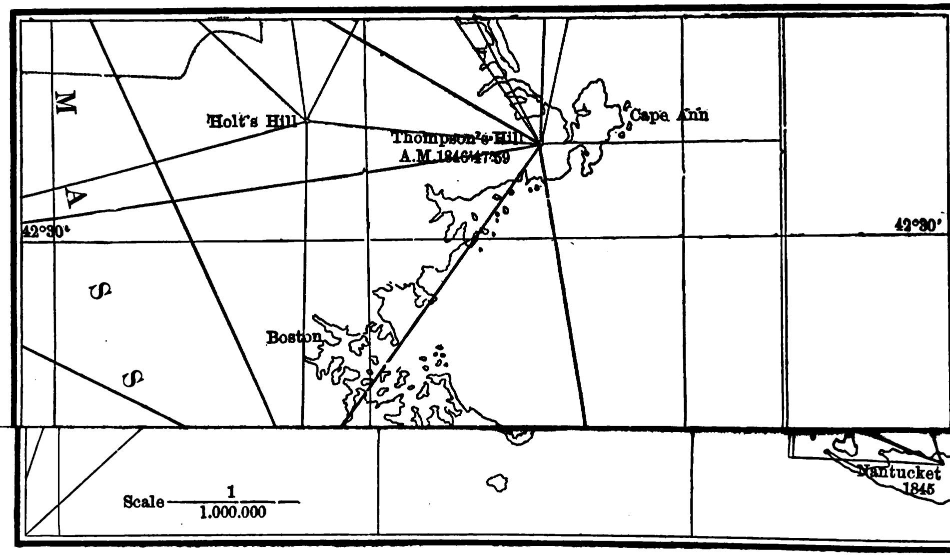
$$q = 5\,130\,740 \text{ toises}.$$

Now it had been established by law that the meter should be one ten-millionth part of the quadrant. Hence

$$1 \text{ meter} = 0.513074 \text{ toises},$$

and, of course, $q = 10\,000\,000$ meters.

23. During the present century the measurement of meridian arcs has generally been carried on only in connection with the triangulations which form the basis of extensive topographical surveys. Central Europe is now covered with a net of triangles, and the same is true



Holt's Hill
Thompson's Hill
A.M. 1846'47'59
Cape Ann
42°30'
42°30'
M
A
S
S
Boston
Nantucket
1845
Scale 1/1.000.000

of portions of Russia, India, and the United States. To obtain a general idea of the processes involved in such work, let us consider for a few moments a portion of the triangulation executed by the United States Coast Survey in New England, and which has furnished a meridian arc of about 3° 23′ in amplitude, or about 233 miles in length. Figure 6 shows the triangulation around the southern half of this meridian arc.* Near the north-eastern corner of the State of Rhode Island you see a line called the Massachusetts base, which was measured along the track of the Boston and Providence Railroad in 1844, with a base apparatus consisting of four bars placed in contact with each other in a wooden box, and provided with micrometer microscopes by which wires could be brought into optical contact with the ends of the bars and with eight thermometers to ascertain the temperature.† The length of the base line is nearly $10\frac{3}{4}$ miles, and its measurement occupied about three months, the exact result corrected for temperature, inclination, and elevation above mean ocean level being 17 326.376 meters, with a probable error of 0.036 meter. About 295 miles north-easterly is the Epping base, and 230 south-westerly is the Fire Island base, which have also been measured with the same careful attention. From the comparison of the measured lengths of these base lines with their lengths as computed through the triangulation, we extract the following values of the Massachusetts base : ‡

* Fig. 6 is a copy of a portion of Sketch No. 3 in the *U. S. Coast Survey Report* for 1875.

† For a complete description, see *Transactions Amer. Phil. Soc.*, 1825, p. 273.

‡ *U. S. Coast Survey Report* for 1865, p. 192.

Measured length	17 326.376	meters.
Calculated from Epping base	17 326.528	"
Calculated from Fire Island base	17 326.445	"

This shows the great accuracy of the work, since the differences of these results exhibit the accumulated errors of all the angle work between the bases as well as those of the linear measurements. The map also shows how from the base line the position of Beaconpole Hill is determined, then that of Great Meadow Hill, and how from these two the triangulation is extended to Blue Hill, and thence onward in all directions. To select proper stations a careful reconnoissance is first made, the tripods and signals are erected, and then there is placed over each station in succession a large and accurate theodolite with which a skilled observer measures all the horizontal angles, each being taken many times on different parts of the arc and in different positions of the telescope, so as to eliminate the instrumental errors. At some of the stations too, astronomical theodolites are placed to determine, by observations on circumpolar stars, the meridian and thence the azimuths of the sides of the triangles; and to find the latitudes a portable zenith telescope is used to measure the difference of the zenith distances of many carefully selected pairs of stars. Longitudes of some of the points are found by comparing with the electric telegraph the local times with that of some established observatory. From these stations of the larger or primary triangles there are formed smaller or secondary triangles, from which the plane table surveys and other topographical work extend out all along the coast line. But before the charts can be published, a great deal of computation is necessary. The observed

angles must be adjusted by the method of least squares, so as to balance in the most advantageous way the small irregular errors of observation. From the bases the lengths of all the sides of the triangles are found, the spherical excesses computed, the adjustments made, and, finally, the latitudes and longitudes of all the stations determined. If now a chain of triangles runs approximately north and south for some distance, these calculations can be readily extended so as to deduce a meridian arc. In Fig. 6 you will notice two parallel lines drawn through the station at Shootflying Hill. This is a part of the meridian arc of 3° 23′ mentioned above. Its southern extremity is in the latitude of Nantucket, and its northern in that of Farmington, Maine. You can see in the map the broken lines drawn perpendicular to the meridian from the several stations; the portions intercepted between these perpendiculars are the meridian distances corresponding to the differences of latitude. The following are the numerical results:

Stations.	Observed Astronomical Latitudes.	Distances between parallels.
		Meters.
Farmington,	44° 40′ 12″.06	58 567.41
Sebattis,	44 8 37 .60	42 718.32
Mt. Independence,	43 45 34 .43	59 535.58
Agamenticus,	43 13 24 .98	67 971.93
Thompson,	42 36 38 .28	76 002.37
Manomet,	41 55 35 .33	70 429.77
Nantucket,	41 17 32 .86	

The total length of the arc is 375 225.38 meters, with a probable error of 1.3 meters. The probable error of an observed astronomical latitude does not exceed 0.1 second. From the whole arc the length of one minute

is found to be 1851.6 meters, with a probable error of 0.6 meter, and the length of one degree in the middle latitude of the arc is 111 096 meters, with an uncertainty of 36 meters.*

24. It is impossible to regard attentively these accurate measures, without a feeling of wonder at the marvelous growth of geodetic science during the present century, not only in instrumental precision, but in theoretical methods of computation. A hundred years ago, for instance, the measurement of the angles of geodetic triangles was so rude that the spherical excess remained undetected, and the processes of adjustment by the method of least squares were entirely unknown. The zenith telescope for latitude observations, the electric telegraph for longitude determination, the self-compensating base apparatus, the method of repetitions in angle measurement, the comparison of the precision of observations by their probable errors, and their adjustment by minimum squares, the theory of spheroidal geodesy —all these and many other improvements have been introduced and perfected in the present century, almost within the memory of men now living.

25. We have explained above a method by which the size of the earth, regarded as an oblate spheroid, may be found by the combination of two measured parts of meridian arcs, and we have also said that at the year 1760, or thereabouts, such combinations of several arcs, taken two by two, gave discordant values for the ellipticity and the length of the quadrant, and that hence it

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* For a full exposition of the methods of calculation, see SCHOTT's valuable report on this arc in *U. S. Coast Survey Report* for 1868, p. 147.

became evident, that either the earth's meridian section was not an ellipse, or that the measurements had not been accurately made. Toward the end of the last century, many attempts were made at rational combinations of the accumulating data, the most important, perhaps, being one by BOSCOVICH in 1760, and two by LAPLACE published in 1793 and 1799 respectively. In order to obtain a clear idea of the problem, let us state the very data used by LAPLACE in his first discussion.

No.	Locality of arc.	Middle latitude.	Length of one degree.
			Toises.
1	Lapland,	66° 20′	57 405
2	Holland,	52 4	57 145
3	France,	49 23	57 074½
4	Austria,	48 43	57 086
5	France,	45 43	57 034
6	Italy,	43 1	56 979
7	Pennsylvania,	39 12	56 888
8	Peru,	0 0	56 753
9	Cape of Good Hope,	33 18	57 037

The numbers in the last column are found by dividing the linear length of each arc in toises by its amplitude in degrees. Now if we consider these short lengths as arcs of a circle, they are directly proportional to the lengths of the radii at their middle points, or if d be the length of any degree and r the radius of curvature of its middle point, evidently

$$d = \frac{2\pi r}{360}.$$

Place in this the value of r from paragraph 19, and it becomes

$$d = \frac{\pi a}{180}(1 - e^2)(1 - e^2 \sin^2 l)^{-\frac{3}{2}}$$

By developing the last factor of this according to the binomial rule, it may be written

$$d = \frac{\pi a}{180}(1 - e^2)(1 + \tfrac{3}{2}e^2 \sin^2 l + \tfrac{15}{8}e^4 \sin^4 l + \ldots).$$

It thus appears that the length of a degree can be expressed by an equation of the form

$$d = \text{M} + \text{N} \sin^2 l + \text{P} \sin^4 l + \ldots\ldots$$

in which

$$\text{M} = \frac{\pi a(1 - e^2)}{180}$$

$$\text{N} = \tfrac{3}{2}e^2\text{M},$$

$$\text{P} = \tfrac{15}{8}e^4\text{M}, \ldots \text{etc.},$$

and LAPLACE, in discussing the above data, considered that it was unnecessary to include powers of e higher than the square, and hence that

$$d = \text{M} + \text{N} \sin^2 l,$$

expressed the length of one degree of the meridian ellipse. Now the problem is this: to deduce from the above seven meridian arcs the values of M and N, so as to obtain an expression for d, the length of one degree of latitude at the latitude l, and then from these values of M and N to find a and e, and all the other elements of the spheroid. And the first step must be to insert

in the formula, the values of d and l for each of the arcs. Thus for arc No. 1,

$$\begin{aligned} d &= 57405 \text{ toises}, \\ l &= 66^\circ\ 20', \\ \sin l &= 0.93565, \\ \sin^2 l &= 0.83887, \end{aligned}$$

and
$$57\,405 = \mathrm{M} + 0.83887\mathrm{N}.$$

In this manner we form the nine following equations:

$$\begin{aligned} 57\,405 &= \mathrm{M} + 0.83887\mathrm{N} \\ 57\,145 &= \mathrm{M} + 0.62209\mathrm{N} \\ 57\,074\tfrac{1}{2} &= \mathrm{M} + 0.57621\mathrm{N} \\ 57\,086 &= \mathrm{M} + 0.56469\mathrm{N} \\ 57\,034 &= \mathrm{M} + 0.51251\mathrm{N} \\ 56\,979 &= \mathrm{M} + 0.46541\mathrm{N} \\ 56\,888 &= \mathrm{M} + 0.39946\mathrm{N} \\ 56\,753 &= \mathrm{M} + 0.00000\mathrm{N} \\ 57\,037 &= \mathrm{M} + 0.30143\mathrm{N} \end{aligned}$$

and from them the values of M and N are to be determined. Now two equations are sufficient to find two unknown quantities, and hence it seems that no values of M and N can be given that will exactly satisfy all of the nine equations. The best that can be done is to find such values as will satisfy them in the most reasonable manner, or with the least discrepancies. To make this idea more definite, suppose that the second members of these equations be transposed to the first, giving equations of the form

$$d - \mathrm{M} - \mathrm{N}\sin^2 l = 0,$$

then since M and N cannot exactly reduce them to zero, we may write

$$
\begin{array}{l}
57\,405 \;\; - \mathrm{M} - 0.83887\mathrm{N} = x_1 \\
57\,145 \;\; - \mathrm{M} - 0.62209\mathrm{N} = x_2 \\
57\,074\tfrac{1}{2} - \mathrm{M} - 0.57621\mathrm{N} = x_3 \\
\text{etc.,} \qquad \text{etc.,} \qquad \text{etc.,}
\end{array}
$$

in which x_1, x_2, x_3, etc., are small errors or residuals. Now LAPLACE, following the idea of BOSCOVICH, conceived that the most reasonable values of M and N were those which would render the algebraic sum of the errors, x_1, x_2, x_3, etc., equal to zero, and also make the sum of the same errors, all taken with the plus sign, a minimum. By introducing these two conditions, he was able to reduce the nine equations to two, from which he found

$$
\begin{aligned}
\mathrm{M} &= 56\,753 \text{ toises}, \\
\mathrm{N} &= 613.1 \text{ toises}.
\end{aligned}
$$

His value of the length, in toises, of one degree of the meridian at the latitude l, was hence

$$d = 56\,753 + 613.1 \sin^2 l.$$

From the values of M and N, it was now easy to find the ellipticity f. Thus, from the above definitions of M and N, we have

$$\frac{\mathrm{N}}{\mathrm{M}} = \frac{3}{2}e^2,$$

from which

$$e^2 = \frac{2 \times 613.1}{3 \times 56\,753} = 0.007202,$$

and then

$$f = 0.0036 = \frac{1}{278}.$$

From the expression for either M or N it is also easy to find a the semi-major axis, whence b, the semi-minor axis, and q, the quadrant of the ellipse, become known.

The last step in LAPLACE's investigation is the comparison of the observed values of the lengths of some of the degrees with those found from his formula for d. For the Lapland arc, for instance, observation gives

$$d = 57\,405 \text{ toises},$$

while computation gives

$$d = 56\,753 + 613.1 \sin^2 66^\circ\, 20' = 57\,267.3,$$

the difference, or error, being

$$137.7 \text{ toises},$$

a distance equal to about 268 meters, or nearly 9 seconds of latitude. These errors, says LAPLACE, are too great to be admitted, and it must be concluded that the earth deviates materially from an elliptical figure.*

26. At the beginning of the present century it was the prevailing opinion among scientists, founded on investigations similar to that of LAPLACE, that the contradictions in the data derived from meridian arcs, when combined on the hypothesis of an oblate spheroidal surface, could not be attributed to the inaccuracies of surveys, but must be due in part, at least, to deviations of the earth's figure from the assumed form. This conclusion, although founded on data furnished by surveys that would nowadays be considered rude, has been confirmed by all later investigations, so that it can be laid down as a demonstrated fact that this earth is not an oblate spheroid. And yet it must never be forgotten that the actual deviations from that form are very small when compared with the great size of the globe itself.

* See *Hist. Acad. Paris* for 1789, pp. 18–43.

In fully half the practical problems into which the shape of the earth enters, it is sufficient to consider it a sphere; in others its variation from a spherical form must be noticed, and there we regard it as spheroidal; cases where it would be requisite to regard its deviation from the spheroidal form will, perhaps, rarely occur in any engineering question; yet for the sake of science we feel curious to determine the laws governing it, and these may at some future time be determined. Now, in the early part of the present century it was agreed by all, notwithstanding the discrepancies of measurements, that for the practical purposes of mathematical geography and geodesy it was highly desirable to determine the elements of an ellipse agreeing as closely as possible with the actual meridian section of the earth. Hence various methods of combination were tried, and as new data accumulated, they were quickly added to the store already on hand, crowding out, gradually to be sure, the older data of less accurate surveys. The most important one of these methods of combination, which is the one now exclusively used for the discussion of precise measurements, was the method of least squares —and a few words must be said concerning its history and explanatory of its processes.

27. In the year 1805 LEGENDRE announced a process for the adjustment of observations, founded upon the principle that the sum of the squares of the residual errors should be made a minimum, and which he named "method of least squares." He gave no proof of the advantage of the principle, but stated it merely as one which seemed to him to be the simplest and the most general, and to secure the most plausible balancing of errors of observation. He deduced some practical rules

for its use and applied it to a numerical example which, it is interesting to observe, was a discussion of the earth's elliptic meridian as resulting from five portions of the long French arc. But in 1809 GAUSS published a theoretical investigation in which he showed from the theory of probability that this method gave the most probable results of the quantities sought to be determined, provided that the observations were subject only to accidental errors—that is, to errors governed by no laws but those of chance. This proof caused the method to be immediately accepted by mathematicians as the only rational process for the adjustment of measurements, and in the following quarter of a century it was fully developed by the labors of GAUSS, BESSEL and others. And here it should not be forgotten that in our own country and in the year 1808, one year in advance of GAUSS, ADRIAN published a proof of the same principle, which unfortunately remained unknown to mathematicians for more than sixty years.* To BESSEL is due the first idea of the comparison of the accuracy of observations by their probable errors, and also many valuable applications of the method to geodetic measurements. It has been truly said that the method of least squares is "the most valuable arithmetical process that has been invoked to aid the progress of the exact sciences;" for the values deduced by it are those which have the greatest probability. With the aid of the theory of probable error the precision of the observations is readily inferred, and uniformity is secured in processes of adjustment and comparison.

* See a paper by ABBE in *Amer. Jour. Sci.*, 1871, vol. i, p. 411. Also List of writings relating to the method of least squares, in *Transactions Conn. Acad.*, 1877, vol. iv, pp. 151–232. Also *Analyst*, 1877, vol. iv, p. 140.

28. To explain the operation of the method, or rather one of its most commonly used operations, let us take a numerical example, and let it be a problem relating to the determination of the earth's ellipticity by pendulum experiments. The following are the data—thirteen values of the length of a seconds pendulum in various parts of the earth as observed by SABINE in the years 1822–24:

Place.	Latitude.	Length of seconds pendulum.
		English inches.
Spitzbergen,	+79° 49′ 58″	39.21469
Greenland,	74 32 19	39.20335
Hammerfest,	70 40 5	39.19519
Drontheim,	63 25 54	39.17456
London,	51 31 8	39.13929
New York,	40 42 43	39.10168
Jamaica,	17 56 7	39.03510
Trinidad,	10 38 56	39.01884
Sierre Leone,	8 29 28	39.01997
St. Thomas,	0 24 41	39.02074
Maranham,	−2 31 43	39.01214
Ascension,	7 55 48	39.02410
Bahia,	12 59 21	39.02425

The ellipticity of the earth may be derived from these observations by means of a remarkable theorem published by CLAIRAUT in 1743, namely,

$$\frac{g}{G} = 1 + \left(\tfrac{5}{2}k - f\right)\sin^2 l,$$

in which G is the force of gravity at the equator, g that at the latitude l, k the ratio of the centrifugal force at the equator to gravity, and f the ellipticity of the earth regarded as an oblate spheroid. This theorem is limited only by the conditions that the form of the earth is

a spheroid of equilibrium assumed in the rotation on its axis, and that its material is homogeneous in each spheroidal stratum. Now, the length of a pendulum beating seconds is proportional to the force of gravity, hence if S represent the length of such a pendulum at the equator, and s the length at the latitude l, the theorem may be also written

$$\frac{s}{S} = 1 + (\tfrac{5}{2}k - f)\sin^2 l.$$

We see then that

$$s = S + T\sin^2 l,$$

in which

$$T = S\,(\tfrac{5}{2}k - f),$$

is a general expression for the length of a second's pendulum. When T and S have been found, their ratio gives the value of $\frac{5}{2}k - f$, and then f the ellipticity becomes known, since k is easily determined with an error of less than half a unit in its third significant figure; (see text-books on mechanics* or astronomy for a proof that $k = \frac{1}{289}$). For each one of the above observations we next write an observation equation, by substituting for s and l their values in the formula

$$s = S + T\sin^2 l.$$

Thus, for the first

$$\begin{aligned} s &= 39.21469 \\ l &= 79^\circ\ 49'\ 58'' \\ \sin l &= 0.9842665 \\ \sin^2 l &= 0.9688402 \\ 39.21469 &= S + 0.9688402T \end{aligned}$$

* WOOD'S *Elementary Mechanics* (New York, 1878), p. 228.

In this manner we find the following thirteen observation equations:

$$
\begin{aligned}
39.21469 &= S + 0.9688402T \\
39.20335 &= S + 0.9289304T \\
39.19519 &= S + 0.8904120T \\
39.17456 &= S + 0.7999544T \\
39.13929 &= S + 0.6127966T \\
39.10168 &= S + 0.4254385T \\
39.03510 &= S + 0.0948286T \\
39.01884 &= S + 0.0341473T \\
39.01997 &= S + 0.0218023T \\
39.02074 &= S + 0.0000515T \\
39.01214 &= S + 0.0019464T \\
39.02410 &= S + 0.0190338T \\
39.02425 &= S + 0.0505201T
\end{aligned}
$$

Now, since the left-hand members of these equations are affected by errors of observations it will not be possible to find values for S and T that will exactly satisfy all the equations; the best that we can do is to find their most probable values, and this is done by the following rule, which may be found proved in all books on the method of least squares:* Deduce a normal equation for S by multiplying each observation equation by the coefficient of S in that equation, and adding the results; deduce also a normal equation for T by multiplying each observation equation by the coefficient of T in that equation and adding the results; thus we shall have two normal equations each containing two unknown quantities, and the solution of these equations will give us the most probable values of S and T. In this case the co-

* See MERRIMAN'S *Elements of the Method of Least Squares* (New York, 1877), p. 155.

efficient of S in each of the equations is unity, multiplying each equation by unity leaves it unchanged, and we have simply to take their sum to get the first normal equation,

$$508.18390 = 13S + 4.8487021T.$$

To find the second normal equation we multiply the first observation equation by 0.9688402, the second by 0.9289304, and so on, and by addition of these results we have

$$189.944469 = 4.8487021S + 3.7043941T.$$

The solution of these two normal equations gives

$$S = 39.01568 \text{ inches,}$$
$$T = 0.20213 \quad \text{"}$$

as the most probable values that can be deduced from the thirteen observations. Hence the length of the seconds pendulum at any latitude l may be written

$$s = 39.01568 + 0.20213 \sin^2 l.$$

Lastly, we find the ellipticity of the earth by the formula

$$f = \frac{5}{2}k - \frac{T}{S},$$

whence

$$f = 0.0086505 - 0.0051807,$$

or

$$f = \frac{1}{288.2}.$$

Before leaving the subject of the pendulum, which we have been obliged to treat very briefly, we will mention that numerous observations of this kind have been made

in various parts of the earth, and that the mean value of the ellipticity deduced from them is $\frac{1}{288.5}$.*

29. During the present century there have been published many investigations and combinations by the method of least squares of the data furnished by the measurement of meridian arcs. The principal results of the most important of these made on the hypothesis of a spheroidal figure are given in the following table: †

Year.	By whom.	Ellipticity.	Quadrant in Meters.
1819	Walbeck	1:302.8	10 000 268
1830	Schmidt	1:297.5	10 000 075
1830	Airy	1:299.3	10 000 976
1841	Bessel	1:299.2	10 000 856
1856	Clarke	1:298.1	10 001 515
1863	Pratt	1:295.3	10 001 924
1866	Clarke	1:295	10 001 887
1868	Fischer	1:288.5	10 001 714
1872	Listing	1:289	10 000 218
1878	Jordan	1:286.5	10 000 681
1880	Clarke	1:293.5	10 001 869

Let us now endeavor to state briefly how such calculations are made. The principle of the method of least squares, it will be remembered, requires that the sum of the squares of the errors of observation shall be rendered a minimum. The first inquiry then is, Where are the errors of observation in a meridian arc—are they in the linear distance, or in the angular amplitude? As long ago as a hundred years, it was suspected that the

* See *Encyclopædia Britannica* (Vol. vii., 1878), Article Earth, p. 608.

† For which I am indebted to Jordan; see his *Handbuch der Vermessungskunde* (1878), Vol. ii., p. 14. The last value of the quadrant has been computed from the data given by Clarke in his *Geodesy* (London, 1880), p. 319.

discrepancies in such surveys were due to deflections of the plumb lines from a vertical, caused by the attraction of mountains, whereby observers were deceived in the position of the zenith and the true level of a station, and hence deduced only apparent or false values of its latitude. It needs indeed not an extensive knowledge of the modern accurate methods of geodesy to become convinced that the errors in the linear distances are very small; on the U. S. Coast Survey, for instance, the probable error in the computed length of any side of the primary triangulation is its $\frac{1}{288000}$th part, which amounts to less than a quarter of an inch in a mile, or two feet in a hundred miles.* The probable error of observation in the latitude of a station is also small, yet it is easy to see that it may be affected with a constant error, due to the deviation of the vertical from the nor-

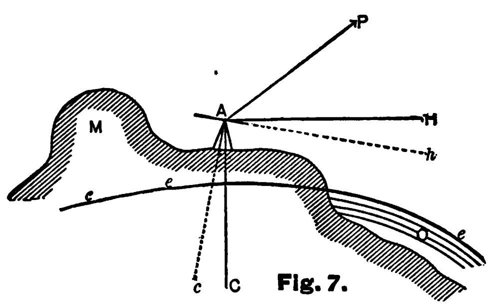


Fig. 7.

mal to the spheroid. To illustrate, let the annexed sketch represent a portion of a meridian section of the earth. O is the ocean, M a mountain, and A a latitude station between them; *eee* is a part of the meridian ellipse coinciding with the ocean surface; AC represents the normal to the ellipse, and AH, perpendicular

* *U. S. Coast Survey Report* for 1865, p. 195.

to AC, the true level for the station A. Now owing to the attraction of the mountain M, the plumb line is drawn southward from the normal to the position Ac, and the apparent level is depressed to Ah. If AP be parallel to the earth's axis, and hence pointing toward the pole, the angle PAH is the latitude of A for the spheroid *eee*; but as the instrument at A can only be set for the level Ah, the observed latitude is PAh, which is greater than the true by the angle HAh. These differences or errors are usually not large—rarely exceeding ten seconds—yet, since a single second of latitude corresponds to about 31 meters or 101 feet, it is evident that the error in the linear distance of a meridian arc is very small, in comparison with that due to a few seconds of error in the difference of latitude. In treating such measurements by the method of least squares we hence regard the distances as without error, and state observation equations which are to be solved by making the sum of the squares of the errors in the latitudes a minimum. Such equations may be stated by writing an expression for the arc of an ellipse in terms of the observed latitudes l_1 and l_2, and measured length of a meridian arc, then in this placing $l_1 + x_1$ and $l_2 + x_2$, instead of l_1 and l_2, the letters x_1 and x_2 denoting the errors in latitude at the stations 1 and 2. The expression will take the form

$$x_2 - x_1 = m + n\mathrm{S} + p\mathrm{T},$$

in which m, n, and p are known functions of the observed quantities, and S and T are known functions of the elements of the ellipse whose values are sought. If there are several latitude stations in a single arc, as is generally the case, one of them should be taken as a reference station, and each error written in terms of the

error there. Thus, if there be four latitude stations, we write

$$\begin{aligned}
x_1 &= x_1 \\
x_2 &= x_1 + m + n\mathrm{S} + p\mathrm{T} \\
x_3 &= x_1 + m' + n'\mathrm{S} + p'\mathrm{T} \\
x_4 &= x_1 + m'' + n''\mathrm{S} + p''\mathrm{T}.
\end{aligned}$$

In like manner there will be a similar series for each arc, each series containing as many equations as there are latitude stations. The first members of these are the latitude errors in regular order, and the sum of the squares of these are to be made a minimum to find the most probable values of S and T; this is done by deriving normal equations for the left-hand members by the usual rule. These normal equations will contain as unknown quantities S and T, and as many errors, x_1, x_5, etc., as there are meridian arcs. When these equations have been solved, it is easy to deduce from S and T the values of the elements of the ellipse. Such, in brief, is the method; but to explain all the details of calculation with the devices for saving labor and insuring accuracy, is not possible here—it would indeed be matter enough for an entire volume.

30. The most important, perhaps, of these discussions is that of BESSEL,* published in 1837, and revised in 1841, because in the meantime an error had been detected in the French survey. We call it the most important, not merely on account of the careful scrutiny given to all the data, and the precise processes of computation employed, but also because its results have been since widely adopted and used in scientific books and geodetic surveys. The material employed by BESSEL consisted

*_Astronomische Nachrichten_, 1841, Vol. xix., p. 97.

of ten meridian arcs—one in Lapland, one each in Russia, Prussia, Denmark, Hanover, England, and France, two in India, and lastly, the one in Peru. The sum of the amplitudes of these arcs is about 50.5°, and they include 38 latitude stations. In the manner briefly described above, there were written 38 observation equations, from which 12 normal equations containing 12 unknown quantities were deduced. The solution of these gave the elements of the meridian ellipse, and also the relative errors in the latitudes due to the deflections of the plumb lines. The greatest of these errors was 6.″45, and the mean value 2.″64. The spheroid resulting from this investigation is often called the BESSEL spheroid, and the elements of the generating ellipse, BESSEL's elements; the values of these will be given below.

31. In 1866 CLARKE, of the British Ordnance Survey, published a valuable discussion, which included a minute comparison of all the standards of measure that had been used in the various countries.* The data were derived from six arcs, situated in Russia, Great Britain, France, India, Peru, and South Africa, including 40 latitude stations, and in total embracing an amplitude of over 76°. This investigation is generally regarded as the most important one of the last quarter of a century, and the values derived by it as more precise than those of BESSEL. The CLARKE spheroid, as it is often called, has been used in some of the geodetic calculations of the United States Coast Survey Office,† and references to it are becoming more frequent in scientific literature every year. It is hence necessary for the student of

* *Comparison of Standards of Length* (London, 1866).
† See *U. S. Coast Survey Report*, 1875, pp. 366–368.

geodesy to be acquainted with the differences between the two spheroids.

32. The following table gives the complete elements of the two spheroids:

	Bessel's Elements. 1841.	Clarke's Elements. 1866.
Semi-major axis a in meters	6 377 397	6 378 206
" " " feet	20 923 597	20 926 062
Semi-minor axis b in meters	6 356 079	6 356 584
" " " feet	20 853 654	20 855 121
Meridian quadrant in meters	10 000 856	10 001 887
Eccentricity e	0.081 697	0.082 271
e^2	0.006 674	0.006 768
Ellipticity f	$\frac{1}{299.15}$	$\frac{1}{294.98}$

From these it is easy to compute the radius of curvature of the ellipses for any latitude, and then to find the lengths of the degrees of latitude and longitude, which are required in the construction of map projections. We give a few of these values in order to exhibit more clearly the differences between the two spheroids:

Latitude.	1 Degree of Latitude on the Spheroid of Bessel.	1 Degree of Latitude on the Spheroid of Clarke.	Difference.
	Meters.	Meters.	Meters.
90°	111 680	111 699	19
50	111 216	111 229	13
45	111 119	111 131	12
40	111 023	111 033	10
35	110 929	110 937	8
30	110 841	110 848	7
25	110 762	110 768	6
0	110 564	110 567	3

For the lengths of the degrees of longitude, there is likewise a difference of about twelve meters in the results deduced from the elements of the two spheroids. For instance, at 50° and at 40° on the BESSEL spheroid we have 71687 and 85384 meters as the lengths of one degree of the parallel, while the corresponding values for the CLARKE spheroid are 71698 and 85396 meters. On a scale of $\frac{1}{10000}$, twelve meters would be 1.2 millimeters; but a sheet of paper exhibiting a whole degree must be several meters in width and length, and hence it would seem that for the practical purposes of map projection the differences between the two spheroids are too small to be generally regarded. The following general values for any latitude l on the CLARKE spheroid may be sometimes useful in computations: The length of one degree of the meridian is

$$364609.87 - 1857.14 \cos 2l + 3.94 \cos 4l,$$

and the length of one degree of longitude is

$$365538.48 \cos l - 310.17 \cos 3l + 0.39 \cos 5l.$$

The radius of curvature of the meridian is

$$20890606.6 - 106411.5 \cos 2l + 225.8 \cos 4l.$$

All these are in feet; to reduce them to meters divide by the number of feet in a meter, namely, 3.28086933, as determined by CLARKE's exact comparisons.

33. Since 1866 the most important contributions to our knowledge of the figure of the earth have been with reference to its ellipsoidal and geoidal forms, rather than to its size considered as a spheroid. In the year just past, however, CLARKE has published a rediscussion

of the data* and deduces the elements of an oblate spheroid that will best satisfy them. The data include the meridian arc of 22° 10′ deduced from the triangulation extending over Great Britain and France, the Russian arc of 25° 20′, one at the Cape of Good Hope of 4° 27′, the Indian arc of 23° 49′, and the Peruvian arc of 3° 6′, making a total of nearly eighty degrees in amplitude. In all these are 56 latitude stations, and the same number of observation equations whose solution by the method of least squares gives

$$a = 20926202 \text{ feet,}$$
$$b = 20854895 \quad \text{``}$$

and for the ellipticity

$$f = \frac{1}{293.465}.$$

The probable error of a single latitude is found to be 1″.645, and the probable error of the number 293.465 about 1.0. From these it is easy to compute

$$q = 10\,001\,869 \text{ meters.}$$

For the length of a degree of latitude at any latitude l he finds

$$364609.12 - 1866.72 \cos 2l + 3.98 \cos 4l,$$

and for the length of the degree of longitude

$$365542.52 \cos l - 311.80 \cos 3l + 0.40 \cos 5l,$$

both being expressed in feet.

---

* CLARKE'S *Geodesy* (Oxford, 1880), pp. 302–322.

34. The form of the earth may also be deduced from measured arcs of parallels between points whose longitudes are known, but as yet geodetic surveys have not furnished sufficient material to effect satisfactory discussions. It is evident that such arcs will have a special value in determining whether or not the equator and the parallels are really circles. Regarding the form as spheroidal, the elements may likewise be found from the length of a geodetic line whose end latitudes and azimuths have been observed. Such a line, extending through the Atlantic States from Maine to Georgia, has been deduced from the primary triangulation of the United States Coast Survey, but its data and the results found therefrom have not yet been published. It is stated, however, that its influence upon our knowledge of the figure of the earth is to but slightly increase the dimensions of CLARKE's spheroid of 1866, without appreciably changing his value of the ellipticity.

35. By regarding the figure of the earth as one of equilibrium assumed under the action of forces due to its gravity and rotation when in a homogeneous fluid state, the value of the ellipticity may be computed from purely theoretical considerations. NEWTON deduced in this way the value $f = \frac{1}{230}$, and an investigation by LAPLACE proves that the ellipticity of a homogeneous fluid spheroid revolving about an axis, and whose form does not differ materially from that of a sphere, is equal to five-fourths of the ratio of the centrifugal force at the equator to the gravitative force. As this ratio is known to be $\frac{1}{289}$, the theorem gives $\frac{1}{231}$ for the ellipticity. But as this value is much too great, the conclusion must be that the earth is not homogeneous.*

* For an exhaustive exposition of this branch of the subject see TODHUNTER's *History of Theories of Attraction*, London, 1873.

36. Lastly, the shape of the earth may be found from astronomical observations and calculations. Irregularities in the motion of the moon were at first explained by the deviation of the earth from a spherical form, and then by precise measurement of the extent of the irregularities, the ellipticity was computed, the value determined by AIRY being $\frac{1}{297}$. As the precession of the equinoxes is due to the attraction of the sun and moon on the excess of matter around the earth's equator, it would seem as if the figure of the globe might be found from that phenomenon also.

37. Looking back now over the historical facts, as here so briefly presented, we may observe that the values of the ellipticity f and of the length of the quadrant q, as deduced from geodetic surveys, have both exhibited a tendency to increase as the data derived from such surveys have become more precise and numerous. About the year 1805 their values, as adopted in the celebrated work for the establishment of the meter, were

$$f = \frac{1}{334} \qquad q = 10\,000\,000 \text{ meters.}$$

In 1841 the investigation of BESSEL gave

$$f = \frac{1}{299} \qquad q = 10\,000\,856 \text{ meters,}$$

in 1866 CLARKE found

$$f = \frac{1}{295} \qquad q = 10\,001\,887 \text{ meters,}$$

and in 1880 he found again

$$f = \frac{1}{293} \qquad q = 10\,001\,868 \text{ meters.}$$

In addition to this we should bear in mind that the

combination of numerous pendulum observations gives, with considerable certainty,

$$f = \frac{1}{289},$$

and this value, it seems not improbable to suppose, will be yet still nearer approached when geodetic measures become more widely extended over the surface of the earth. For very many problems it will be found convenient to keep in mind the following round numbers:

$$\text{Ellipticity} = \frac{1}{17^2} = \frac{1}{289}.$$

$$\text{Eccentricity} = \frac{1}{12}.$$

$$\text{Quadrant} = 10001 \text{ kilometers.}$$

The following mnemonic rule may perhaps be of some use in remembering the values of the semi-equatorial and semi-polar diameters a and b: keep in mind the number 6400 kilometers (which is a perfect square), then a is 22 kilometers and b is 44 kilometers less than this. In the form of an equation

$$a = (6400 - 22) \text{ kilometers},$$
$$b = (6400 - 44) \text{ kilometers};$$

and the difference $a - b$ equals 22 kilometers. In English measures there is also the following easily remembered approximate value of the polar axis, namely,

$$2b = 500\,500\,000 \text{ inches.}$$

38. Three hundred and fifty years ago, when men began first to think about the shape of the earth on

which it was their privilege to live, they called it a sphere, and they made rude little measurements on its great surface to ascertain its size. These measurements, as we know, at length after nearly two centuries, reached an extent and precision sufficient to prove that its surface was not spherical. Then the earth was assumed to be a spheroid of revolution, and with the lapse of time the discrepancies in the data, when compared on that hypothesis, proved also that the assumption was incorrect. Granting that the earth is a sphere, there has been found the radius of one representing it more closely than any other sphere; granting that it is a spheroid, there has been also found, from the best existing data combined in the best manner, the dimensions of one that represent it more closely than any other spheroid. But as further and more accurate data accumulate, alterations in these elements are sure to follow. In the last chapter we saw that the radius of the mean sphere could only be found by first knowing the elliptical dimensions, and here it might, perhaps, be also thought that the best determination of the most probable spheroid would be facilitated by some knowledge of the theory of the size and shape of the earth considered under forms and laws more complex than those thus far discussed. In the following chapters, then, we shall endeavor to give some account of the present state of scientific knowledge and opinion concerning the earth as an ellipsoid with three unequal axes, the earth as an ovaloid, and lastly, the earth as a geoid.

CHAPTER III.

THE EARTH AS AN ELLIPSOID.

39. Just as the sphere is a particular case of the spheroid of revolution, so the spheroid of revolution is a particular case of the ellipsoid. The sphere is determined by one dimension, its radius; the spheroid by two, its polar and equatorial diameters; while in the ellipsoid there are three unequal principal axes at right angles to each other, which establish its form and size. Like the spheroid, the ellipsoid has all its meridian sections ellipses; but the equator, instead of being a circle, is an ellipse of slight eccentricity, and its two axes, together with the polar axis of rotation, constitute the three principal diameters. Let a_1 and a_2 denote the greatest and the least semi-diameters of the equator of the ellipsoid, and b the semi-polar diameter. The ellipticity of the greatest meridian ellipse is then

$$f_1 = \frac{a_1 - b}{a_1},$$

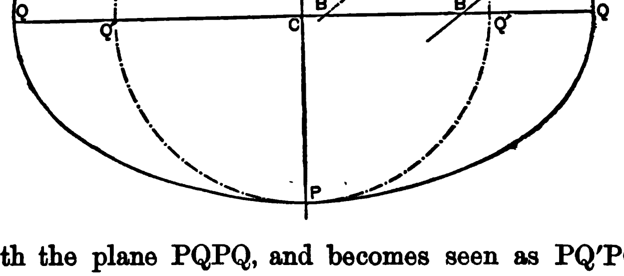
and that of the least is

$$f_2 = \frac{a_2 - b}{a_2},$$

while the ellipticities of all the other meridian ellipses have values intermediate between f_1 and f_2. For the

equator the ellipticity is $\frac{a_1 - a_2}{a_1}$. When the values of a_1, a_2, and b are known, the dimensions and proportions of the meridian ellipses and of all other sections of the ellipsoid can be easily found. In such a figure, however, the curves of latitude, with the exception of the equator, are not plane curves, and hence cannot be called true parallels; this results from the definition of latitude, and may be seen from the following diagram: PP is the polar axis, PQPQ the greatest meridian section of the ellipsoid, and A a place of observation upon it, whose horizon is AH and latitude ABQ, AB being the direction of the plumb line at A, which of course is perpendicular to the tangent horizon line AH. Let now the least meridian ellipse, projected in the line PP, be conceived to revolve around PP until it coincides

Fig. 8



with the plane PQPQ, and becomes seen as PQ'PQ'. To find upon it a point A' that shall have the same latitude as A, it is only necessary to draw a tangent

A'H' parallel to AH touching the ellipse at A', then A'B' perpendicular to A'H' makes the same angle with the plane of the equator QQ as does AB. If the least meridian section be now revolved back to its true position, A' becomes projected at D'. We thus see, that, while a section through A parallel to the equator is an ellipse ADA, the curve joining the points having the same latitude as A is not a plane curve, but a tortuous line, AD'A.

40. The process for determining from meridian arcs an ellipsoid to represent the figure of the earth, does not differ in its fundamental idea from that explained in the last chapter for the spheroid. The normal to the ellipsoid at any point will usually differ slightly from the actual vertical as indicated by the plumb line, and these deviations are taken as the residual errors to be equalized by the method of least squares. An expression for the difference of these deviations at two stations on the same meridian arc is first deduced in terms of four unknown quantities, three being the semi-axes a_1, a_2, and b, or suitable functions of them, and the fourth the longitude of the greatest meridian ellipse, referred to a standard meridian such as that of Greenwich; and in terms of four known quantities, the observed linear distance between the two stations, their latitudes and the longitude of the arc itself. Selecting now one station in each meridian arc as a point of reference, we write for that arc as many equations as there are latitude stations, inserting the numerical values of the observed quantities. These equations will contain four more unknown letters than there are meridian arcs, and from them by the method of least squares as many normal equations are to be deduced as

there are unknown quantities, and the solution of these will furnish the most probable values of the semi-axes a_1, a_2, and b, with the longitude of the extremity of a_1, and also the probable plumb-line deviations at the standard reference stations. The process is long and tedious, but it is easy to arrange a system and schedule, so that, starting with the data, most of the labor may be done by computers, who have no idea at all of the whys and wherefores involved, and who work for pay and not for science.

41. The first deduction of an ellipsoid to represent the figure of the earth was made in Russia, by SCHUBERT, about the year 1859. His data consisted of eight meridian arcs, the Russian, English, Prussian, French, Pennsylvanian, Indian, Peruvian, and South African, embracing in total an amplitude of about 72°. These were combined in a manner different and less satisfactory than that above described, the results, according to LISTING,* being,

$$a_1 = 6\ 378\ 556 \text{ meters.}$$
$$a_2 = 6\ 377\ 837 \quad \text{"}$$
$$b = 6\ 356\ 719 \quad \text{"}$$
$$q_1 = 10\ 002\ 263 \quad \text{"} \qquad f_1 = \frac{1}{292.1}$$
$$q_2 = 10\ 001\ 707 \quad \text{"} \qquad f_2 = \frac{1}{302.0}$$
$$Q = 10\ 018\ 849 \quad \text{"} \qquad F = \frac{1}{8881}$$

Longitude of $q_1 = 40°\ 37'$ E. of Greenwich.

---

* LISTING, *Unsere jetzige Kenntniss der Gestalt und Grösse der Erde* (Göttingen, 1873), p. 35.

Here q_1 and q_2 are the quadrants of the greatest and least meridian ellipses, Q the quadrant of the equator, and f_1, f_2, and F the corresponding ellipticities; a_1 and a_2 are the equatorial semi-axes, and b the polar semi-axis. By referring to a map of the earth we see that the maximum meridian ellipse passes through Russia and Arabia in the eastern continent, and through Alaska and the Sandwich Islands in the western, while the minimum ellipse cuts Japan, Australia, Greenland, and South America.

42. It is, however, CLARKE, of the British Ordnance Survey, to whom we owe almost all our knowledge of the dimensions of the earth as an ellipsoid. His first investigation was made in 1860, and embraced the data from the Russian, English, French, Indian, Peruvian, and South African arcs, in all more than five-sixths of a quadrant, and containing 40 latitude stations. This calculation was revised in 1866, on account of slight changes in the data due to a careful comparison of the different standards of measure, and gave the following results as the most probable elements of the spheroid:

$$
\begin{array}{llll}
a_1 = & 6\,378\,294 \text{ meters} & = 20\,926\,350 \text{ feet.} \\
a_2 = & 6\,376\,350 \quad `` & = 20\,919\,972 \quad `` \\
b = & 6\,356\,068 \quad `` & = 20\,853\,429 \quad `` \\
q_1 = & 10\,001\,553 \quad `` & f_1 = \frac{1}{287.0} \\
q_2 = & 10\,000\,024 \quad `` & f_2 = \frac{1}{314.4} \\
Q = & 10\,017\,475 \quad `` & F = \frac{1}{3281}
\end{array}
$$

Longitude of $q_1 = 15° \; 34'$ East.

The equator is here more elliptical than in SCHUBERT'S ellipsoid, while the greatest meridian lies 25° farther west, passing through Scandinavia, Germany, Italy, Africa, the Pacific Ocean, and Behring's Straits. The least meridian coincides nearly with that of New York. The data entering these elements are the same as for the CLARKE spheroid of 1866; in fact, by a slight change in the equations, equivalent to making $a_1 = a_2 = a$, the ellipsoid may be rendered a spheroid, and the elements of the latter also deduced.

43. In 1878, CLARKE published * the results of a third discussion, in which the above-described data were augmented by a new meridian arc of 20° in India and by several arcs of longitude. The solution of 51 equations gave the following:

$$
\begin{array}{llll}
a_1 = & 6\ 378\ 209 \text{ meters} & = 20\ 926\ 629 \text{ feet.} \\
a_2 = & 6\ 376\ 202 \quad \text{"} & = 20\ 925\ 105 \quad \text{"} \\
b = & 6\ 356\ 076 \quad \text{"} & = 20\ 854\ 477 \quad \text{"} \\
q_1 = & 10\ 001\ 867 \quad \text{"} & f_1 = \dfrac{1}{290} \\
q_2 = & 10\ 001\ 507 \quad \text{"} & f_2 = \dfrac{1}{296.3} \\
Q = & 10\ 018\ 770 \quad \text{"} & F = \dfrac{1}{13706}
\end{array}
$$

Longitude of $q_1 = 8°\ 15'$ West.

The equator is here less elliptical. The greatest meridian passes through Ireland, extreme western Africa, through New Zealand and the north-east corner of Asia, while the least meridian passes through Central

* *Philosophical Magazine*, August, 1878.

Asia and Central North America. These meridians are remarkably situated with reference to the physical features of the globe.

44. At the present time it seems to be the prevailing opinion that satisfactory elements of an ellipsoid to represent the earth cannot be obtained until geodetic surveys shall have furnished more and better data than are now available, and particularly data from arcs of longitude. The ellipticities of the meridians differ so slightly that measurements in their direction alone will, probably, be insufficient to determine, with much precision, the form of the equator and parallels. In Europe, several longitude arcs will soon be available, and, perhaps, fifty years hence the primary triangulation of our Coast and Geodetic Survey may extend from the Atlantic to the Pacific. If it then be thought desirable to represent the earth by an ellipsoid with three unequal axes rather than by a spheroid, its elements can be determined with some satisfaction. At present the ellipsoids represent the figure of the earth as a whole very little better than do the spheroids, although, for certain small portions, they may have a closer accordance. For instance, the average probable error of a plumb-line deviation from the normals to the CLARKE ellipsoid of 1866 is $1''.35$, while for the spheroid derived from the same data it is $1''.42$. Further, the marked differences in the ellipticities of the equator of the two CLARKE ellipsoids, due to comparatively slight differences in data, are not pleasant to observe. And, lastly, the ellipsoid is a more inconvenient figure to use in calculations than the spheroid. For these reasons the earth has not yet been regarded as an ellipsoid in prac-

tical geodetic computations, and it is not probable that it will be for a long time to come.*

* In his work on Geodesy published in 1880, CLARKE says, referring to his ellipsoidal investigation of 1878: "Although the Indian longitudes are much better represented than by a surface of revolution, it is necessary to guard against an impression that the figure of the equator is thus definitely fixed, for the available data are far too slender to warrant such a conclusion."

CHAPTER IV.

THE EARTH AS AN OVALOID.

45. In a spherical, spheroidal, or ellipsoidal earth the northern and southern hemispheres are symmetrical and equal; that is to say, a plane parallel to the equator, at any south latitude, cuts from the earth a figure exactly equal and similar to that made by such a plane at the same north latitude. The reasons for assuming this symmetry seem to have been three: first, a conviction that a homogeneous fluid globe, and hence perhaps the surface of the waters of the earth, must assume such a form under the action of centrifugal and centripetal forces; secondly, ignorance and doubt of any causes that would tend to make the hemispheres unequal; and thirdly, an inclination to adopt the simplest figure, so that the labor of investigation and calculation might be rendered as easy as possible. The first of these is perhaps an excellent reason, considered by itself alone; but when we begin to speculate about the probability of any regular law in the density of the earth, and further, when we find plumb-line deviations only to be reconciled on supposition of non-homogeneity, it seems to assume more the nature of a rough analogy. The last is a perfectly proper reason when viewed from an engineering point of view, for where practical calculations are to be made they should be so conducted that the desired results may be obtained at

a minimum cost; and this argument will always, more or less, affect even the most abstruse scientists in whose investigations there is perhaps no thought of practical utility. The second reason is not so valid to-day as it was a century ago, for gradually there have come into men's minds a great many thoughts which now lead us to suppose that there are several causes that tend to make the southern hemisphere greater than the northern. These thoughts embrace a vast field of inquiry and speculation in astronomy, physics, and geology; but we can here only briefly hint at two or three of the principal facts and conclusions.

46. The earth moves each year in an ellipse, the sun being in one of the foci, and revolves each day about an axis, inclined some $66\frac{1}{2}°$ to the plane of that orbit. When this axis is perpendicular to a line drawn from the center of the sun to that of the earth occur the vernal and autumnal equinoxes, and at points equally removed from these are the summer and winter solstices. For many centuries the earth's orbit has been so situated in the ecliptic plane that the perihelion, or nearest point to the sun, has nearly coincided with the winter solstice of the northern hemisphere and the summer solstice of the southern hemisphere. The consequences are: first, that the half of the year corresponding to the winter, is about seven days longer in the southern hemisphere than in the northern; secondly, that during the year the south pole has about 170 more hours of night than of day, while the north has about 170 more hours of day than of night; and, thirdly, that winter in the northern hemisphere occurs when the sun is at his least distance from the earth, and in the southern when he is at his great-

est. From these three reasons it would seem that the amounts of heat at present annually received by the two hemispheres should be unequal, the northern having the most and the southern the least. Now, when we glance at the geography and meteorology of the globe, these two facts are seen: first, that fully three-fourths of the land is in the northern hemisphere clustered about the north pole, while the waters are collected in the southern; and secondly, that the south pole is enveloped and surrounded by ice to a far greater extent than the northern. There is then a considerable degree of probability that some connection exists between these astronomical and terrestrial phenomena, that the former, indeed, may be the cause of the latter. The mean annual temperature of the earth's southern hemisphere during so many centuries may have been enough lower than the northern to have caused an accumulation of ice and snow whose attraction is sufficient to drag the waters toward it, thus leaving dry the northern lands and drowning the southern with great oceans. It appears then somewhat probable that there are causes tending to render the earth ovaloidal or egg-like in shape, the large end being at the south and the small at the north.

47. The process of finding the dimensions of an ovaloid of revolution to represent the form of the earth would be essentially the same as that already described for the spheroid and ellipsoid. First, the equation of an oval should be stated and, preferably, one that by the vanishing of a certain constant reduces to an ellipse. From this equation an expression for the length of an arc of north and south latitude can be deduced, and this be finally expressed in terms of the small deviations

between the plumb lines and the normals to the ovaloidal meridian section at the latitude stations. The solution of these equations by the method of least squares will give the most probable values of the constants, determining the size and shape of the oval due to the data employed. Such computations have not yet been undertaken, on account of the lack of sufficient data from geodetic surveys in the southern hemisphere. Since such surveys can only be executed on the continents and largest islands, it is clear that the data will always be few in number compared with those from the northern hemisphere. Pendulum observations, discussed on the hypothesis of a spheroidal globe, by CLAIRAUT'S theorem, are able, however, to give some information concerning it; but, unfortunately, the number of these thus far made south of the equator is not sufficiently large to render them of much value in the investigation. It is probable that in years to come pendulum observations, or other methods for measuring the intensity of gravity, will be more employed than they are at present; and since they can be made on small islands as well as on the main lands, it is possible thereby to obtain knowledge concerning the separate ellipticities of the two hemispheres.

48. An important idea to be noted in this branch of our subject is, that the surface of the waters of the earth is, probably, not fixed but variable. About the year 1250, the perihelion and the northern winter solstice coincided, and the excess in annual heat imparted to the northern hemisphere was near its maximum. Since that date they have been slowly separating, and are now nearly eleven degrees apart. This separation increases annually by about $61''.75$, so that a motion of

180° will require about 10450 years, and when that is accomplished the perihelion will coincide with the southern winter solstice. Then the condition of things will be exactly reversed; the northern hemisphere will receive less heat than the southern, and if to such a degree as some have conjectured, then the ice will accumulate around the north pole, the waters will flow back from the south to the north, the lands in the northern hemisphere become submerged while those in the southern are left dry. The change may be so slow that for many centuries it might remain undetected, and yet ultimately be sufficient to perceptibly alter the relative shapes and sizes of the two hemispheres.* The period of a complete cycle is about 20900 years, so that in the year 22150, of the Gregorian calendar, conditions will exist similar to those in 1250. Long before that time it is not improbable that civilization will disappear and a cloud of intellectual darkness settle over mankind. Possibly enough, too, is it that in that remote age, as in the two centuries following the year 1250, men may waken out of their mental stupor and begin to make feeble inquiries about the size and shape of the earth on which it is their destiny to dwell.

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* For a historical review of opinions concerning these changes see GÜNTHER'S brochure, *Die chronische Versetzung des Erdschwerpunktes durch Wassermassen*, Halle, 1878.

CHAPTER V.

THE EARTH AS A GEOID.

49. THE word Geoid is used to designate the actual figure of the surface of the waters of the earth. The sphere, the spheroid, the ellipsoid, the ovaloid, and many other geometrical figures may be, to a less or greater degree, sufficient practical approximations to the geoidal or earthlike shape, yet no such assumed form can be found to represent it with precision. The geoid, then, is an irregular figure peculiar to our planet; so irregular, indeed, that some have irreverently likened it unto a potato; and yet a figure whose form may be said to be subject to fixed physical laws, if only the fundamental ideas implied in the name be first clearly and mathematically defined.

50. The first definition is, that the surface of the geoid at any point is perpendicular to the direction of the force of gravity, as indicated by the plumb line at that point; from the laws of hydrostatics it is evident that the free surface of all waters in equilibrium must be parallel to that of the geoid. And the second definition determines that our geoidal surface to be investigated is that coinciding with the surface of the great oceans, leaving out of consideration the effects of ebb and flood, currents and climate, wind and weather. Under the continents and islands this surface may be

conceived to be produced so that it shall be at every point perpendicular to the plumb-line directions. If a tunnel be driven exactly on this surface from ocean to ocean it is evident that the water flowing from each would attain equilibrium therein, and its level finally show the form of the geoid along that section of the earth.

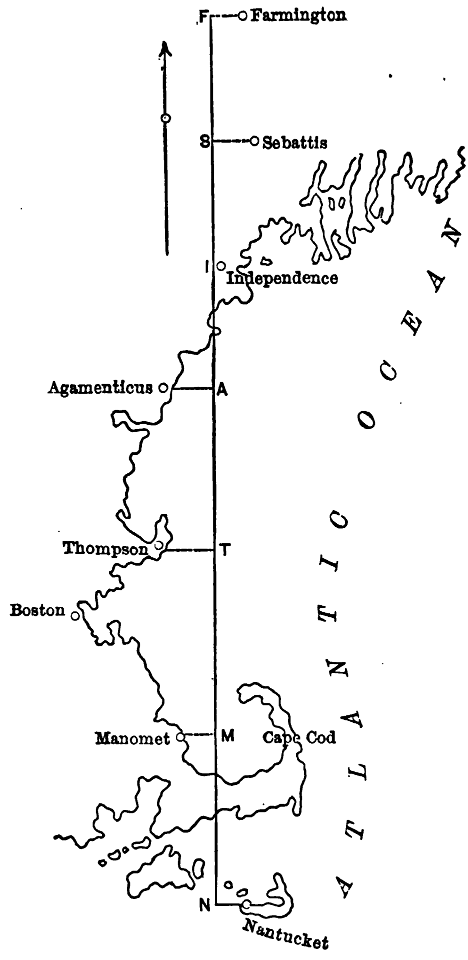
51. To obtain a clearer idea of the properties of the geoid, let us consider again the meridian arc measured by the United States Coast Survey in New England, and particularly the following values of the latitudes at the latitude stations: *

Stations.	Astronomical Latitudes.	Geodetic Latitudes.	Difference.
	° ′ ″	° ′ ″	″
Farmington.	44 40 12.06	44 40 14.31	+ 2.25
Sebattis.	44 8 37.60	44 8 36.68	− 0.92
Independence.	43 45 34.43	43 45 32.47	− 1.96
Agamenticus.	43 13 24.98	43 13 23.16	− 1.82
Thompson.	42 36 38.28	42 36 40.24	+ 1.96
Manomet.	41 55 35.33	41 55 36.77	+ 1.44
Nantucket.	41 17 32.86	41 17 33.66	+ 0.80

The column headed astronomical latitudes contains the values observed—that is, the angles included between the plane of the earth's equator and the plumb-line directions at each point; while the other column contains the geodetic latitudes—that is, the angles included between the plane of the earth's equator and the normals to a Bessel spheroid, as computed by the use of the triangulation. The vertical directions as given by the geodetic latitudes are hence normal to the

* *U. S. Coast Survey Report*, 1868, p. 150.

Fig. 9



spheroid, while those as shown by the observed astronomical latitudes are normal to the geoid. The differences of these two, as noted in the last column, are the

same as the angles between the two normals, and indicate the relative plumb-line deflections at the stations. The above figure shows on a small scale the general trend of the coast, the position of the latitude stations and the meridian arc. It might, perhaps, be expected in advance that the actual directions of the plumb lines at these points would deviate northwestwardly from the normals to a spheroid for two reasons ; first, because of the heavier continent lying north and west, and secondly, because of the lighter waters lying south and east. To judge concerning this, let us imagine a section of the earth and the spheroid and the geoid along the meridian arc. Let F be a point on this meridian having the same latitude as Farmington, S a point having the same latitude as Sebattis, and similarly for the other stations, and let us consider that the plumb-line directions at these points are the same as at the latitude stations themselves, as far at least as north and south deviations are concerned. Draw, as in the following figure 10, an arc of an ellipse FSIATMN to represent a section of the spheroid along the meridian arc, and let the distances FS, SI, etc., be laid off to scale equal to the distances as found from the base line and triangulation. (See paragraph 23.) Draw at these points normals to the ellipse; these will make with the earth's equator (to which QQ in the figure is drawn parallel) angles equal to the above geodetic latitudes. At F draw a line FZ making with QQ an angle equal to the observed astronomical latitude, so that SFX represents $2''.25$, the plumb-line deviation at F. Draw at each of the other points similar broken lines, each of which must indicate the direction of the true zenith Z of its respective station. Now, the surface of the Atlantic Ocean coincides with that of the geoid ; let there, then, be drawn in the

plane of the section a broken curved line perpendicular to the true plumb-line directions to represent this surface, and let it be produced under the continent according to the same law. The figure now exhibits roughly the probable approximate relative positions of the spheroid and the geoid along this meridian arc, and a careful study of it will be advantageous in enabling us to clearly perceive some of the principal properties of the geoidal surface. We observe that under the continents it tends to arise higher, while on the seas it tends to

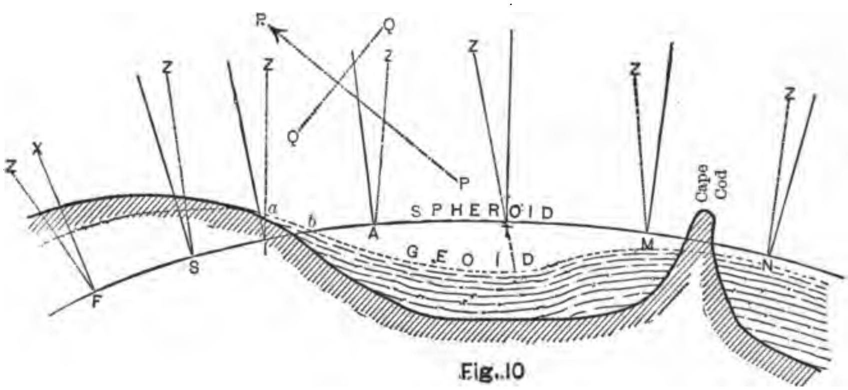


Fig. 10

sink lower than the surface of a spheroid of equal volume. (But probably never is it convex toward the earth's center as indicated in the exaggerated drawing.) The reason of this is easy to see when we regard the geoid as a figure formed under the action of the attractive force of the matter of the globe. The attraction of the heavier and higher continents lifts, so to speak, the geoidal surface upward, while the lower and lighter ocean basins allow it to sink downward toward the earth's center. But the figure also shows that this rule has its exceptions ; the true vertical or plumb-line direc-

tion at Farmington, for instance, inclines to the northward of the zenith of the normal to the spheroid instead of southward, as we perhaps might expect it to do. Such anomalies are, in fact, very frequent, and from them we conclude that the earth's crust is of quite variable density, and that this causes the apparent irregularities in the directions of the force of gravity in neighboring localities.

52. We may also see that what we have called plumb-line deflections are really something artificial, depending upon the use of a particular spheroid. The geoid is an actual existing thing, the spheroid is not, but is largely an assumption introduced for practical and approximate purposes. At the station F, in the above figure 10, the direction FZ is the only one that can be observed, and the angle made by it with QQ has been measured with a probable error of less than one-tenth of a second of arc. The angle ZFX, or the so-called plumb-line deflection at F, will hence vary with the elements of the particular spheroid employed, and with the correct orientation of geoid and spheroid. A geodetic latitude (or spheroidal latitude as it should perhaps be more properly called) is something that cannot be directly measured, and therefore it seems that the plumb-line deviations for even a particular spheroid cannot be absolutely found until observations have been made over an extent of country wide enough to enable us to judge of the laws governing the geoid itself. A very slight change in the position of the above elliptical arc may add or subtract a constant quantity from each of the angles between the true verticals and the normals. The differences of the plumb-line deflections at neighboring stations will, however, always remain the

same. For instance, at T and M the excesses of geodetic over astronomical latitudes are $1''.96$ and $1''.44$, whose difference is $0''.52$; but the spheroid may also be drawn giving $1''.66$ and $1''.14$ for these deviations, and their difference is likewise $0''.52$. Strictly speaking, then, it is not the plumb line which deflects, but it is the normal to an artificial spheroid or ellipsoid which deviates from the constant plumb-line direction.

53. Compared with a spheroid of equal volume, our geoid has a very irregular surface, now rising above that of the spheroid, now falling below it, and ever changing the law of its curvature, so as to conform to the varying intensity and direction of the forces of gravity. Where the earth's crust is of most density and thickness there it rises, where the crust is of least density and thickness there it sinks. From a scientific point of view it will be valuable to know the laws governing its form and size; from a practical point of view it appears that until these are known the earth's figure can never be accurately represented by a sphere or spheroid or ellipsoid, or other geometrical form. For instance, if it be desired to represent the earth by an oblate spheroid, the best and most satisfactory one must be that having an equal volume with the geoid, and whose surface everywhere approaches as nearly as possible to the geoidal surface. Such a spheroid cannot, of course, be found until more and better data concerning the geoid have accumulated, yet what has already been said is sufficient to indicate that the dimensions at present used are probably somewhat too large. Granting that in general the geoid rises above this spheroid under the continents and falls below it on the seas it seems evident, since the area of the oceans is nearly three times

that of the lands, that the intersection of the two surfaces will always be some distance seaward from the coast line (as seen at *b* in Fig. 10). Now geodetic surveys can only be executed on the continents, and even if they be reduced to the sea level at the coast (*a* in Fig. 10), the elements of a spheroid deduced from them will be too large to satisfy the above condition of equality of volumes (for the ellipse through *a* is evidently larger than that through *b*). At present it would be almost a guess to state what quantity should be subtracted from the semi-axes of the CLARKE spheroid on account of these considerations; but there are reasons for thinking that 500 meters would be too much.

54. We have now to briefly consider the important question, how can the shape and size of our geoid, and its position with reference to the earth's axis of rotation, be determined? From what has already been said, it is not difficult to conclude that a fair mental picture of its surface may be acquired for a locality where precise geodetic surveys have been executed. At places along the coast let the sea level be determined as due to the earth's attraction alone, the effect of tides, currents, and storms being eliminated. These are points on the geoidal surface, and it may be imagined to be produced inland, so that everywhere it shall be perpendicular to the direction of the force of gravity. To obtain numerical data regarding its form and position, it may be referred to the surface of a spheroid, the direction and amount of the plumb-line deflections indicating always its change of curvature and its relative elevation or depression as compared with the spheroid. But on the oceans, where geodetic operations cannot be executed, it will, probably, ever be impossible to obtain

such numerical results. At the present time there is very little known regarding the actual figure of the geoid even on the continents. The word Geoid, in fact, with all the fruitful ideas therein implied, is not yet ten years old,* and in all relating to it theory is in advanco of practice. Bruns, for instance, has demonstrated that the mathematical figure of the earth may be determined independently of any hypothetical assumption concerning the law of its formation, provided that there have been observed at and between numerous stations five classes of data, namely, astronomical determinations of latitude, longitude, and azimuth, base line and triangulation measurements, vertical angles between stations, spirit leveling between stations, and determinations of the intensity of the forces of gravity.† These five classes are sufficient for the solution of the problem, but also necessary; that is, if one of them does not exist, a hypothesis must be made concerning the shape of the earth's figure. These complete data have, however, never yet been observed for even an extent of country so small as England, a land probably more thoroughly surveyed than any other. To render geodetic results of the greatest scientific value, it is hence necessary that either the pendulum, or some instrument like Siemens' bathometer, should be employed to determine the relative intensity of the forces of gravity at the principal triangulation stations, and that trigonometric leveling, by vertical angles, should be brought to greater perfection. But years and centuries must roll away before sufficient data shall have accumulated to render a theoretical discussion satisfactory in its results.

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* It was first used by Listing in 1872.

† Bruns, *Die Figur der Erde*, Berlin, 1878.

55. In conclusion, it will be well to note that our geoid is not a fixed and constant figure. Upon the earth men build towns and cause ships and trains to move; simultaneously with these displacements of matter, wrinkles and waves appear in the geoidal surface. But the changes that man can effect are infinitesimal in comparison with those produced by nature. The atmospheric elements are continually at work to tear down the continents and fill up the ocean basins; ever conforming to such alterations the geoid tends to nearer and nearer uniformity of curvature. Internal fires cause parts of the earth's crust to slowly rise or fall, and immediately the geoidal surface undergoes a like alteration. If the center of gravity of the earth oscillates north and south during the long apsidial cycle of 20900 years, the position and shape of the geoid will vary slowly with it. Perhaps also the axis of rotation of the earth may not be invariable with respect to its mass, but subject to slight oscillations. The changes produced by these causes are not all so minute as to escape detection, for already small but measurable variations have been discovered in the latitudes of several of the oldest observatories, and we may expect that in future centuries other alterations still will be noticed and observed and discussed. When the laws governing all these changes shall have become understood, it will be possible to reason more accurately than now concerning the past history and future destiny of our earth.

THE PRINCIPLES

OF

LEAST SQUARES.

THE

PRINCIPLES OF LEAST SQUARES.

SECTION I. ERRORS OF OBSERVATIONS.

WHEN a quantity is measured the result of the operation is a numerical value called an observation. If Z be the true value of a quantity and M_1 and M_2 be two observations upon it, then $Z - M_1$ and $Z - M_2$ are the errors of those observations.

Constant or systematic errors are those which result from causes well understood and which can be computed or eliminated. As such may be mentioned: theoretical errors, like the effects of refraction upon a vertical angle, or the effects of temperature upon a steel tape, which can be computed when proper data are known and hence need not be classed as real errors; instrumental errors, like the effects of an imperfect adjustment of an instrument, which can be removed by taking proper precautions in advance; and personal errors which are due to the habits of the observer, who may, for example, always give the reading of a scale too great. All these causes are to be carefully investigated and the resultant errors removed from the final observations in cases of precise work.

Mistakes are due to mental confusion so serious that the observation cannot be regarded with any confidence. Observations affected with mistakes must be rejected, although when these are of small magnitude it is sometimes not easy to distinguish them from errors.

Accidental errors are those that still remain after all constant errors and mistakes have been carefully investigated and eliminated from the results. Such, for example, are the errors in levelling arising from sudden expansions of the bubble and standards, or from the effects of the wind, or those due to refraction. And also such errors arise from the imperfections of human touch and sight, which render it difficult for us to handle instruments delicately or to read verniers with perfect accuracy. These are the errors that exist in the final observations and whose elimination forms the subject of these pages.

However carefully the observations be made, the final observations do not agree, owing to the fact that the errors have different magnitudes. All of these cannot be correct, since the quantity has only one value, and each of them can be regarded only as an approximation to the truth. The absolutely true value of the quantity in question cannot be ascertained, but instead of it one must be determined, derived from the combination of the observations, which shall be the "most probable value," that is to say, the value which is probably most nearly to the truth. The Method of Least Squares teaches how to derive from observations the most probable values of observed quantities.

The differences between the most probable value of a quantity and the observations are called "residual

errors." If z be the most probable value of a quantity derived from the observations M_1 and M_2, the residual errors are $z - M_1$ and $z - M_2$. When the observations are taken with precision, the most probable value z does not greatly differ from the true value Z, and the residual errors do not greatly differ from the true errors.

It is found by experience that errors of observations are not distributed at random, but that they are governed by law. These laws are: (1) small errors are more frequent than large ones; (2) positive and negative errors are equally numerous; (3) very large errors do not occur. For instance, an examination of 100 observations on angles of the primary triangulation stated on page 91 of the Report of the United States Coast Survey for 1854, shows that the residual errors were distributed as follows:

Between	$+6''.0$	and	$+5''.0$	1 error
Between	$+5\ .0$	and	$+4\ .0$	2 errors
Between	$+4\ .0$	and	$+3\ .0$	2 errors
Between	$+3\ .0$	and	$+2\ .0$	3 errors
Between	$+2\ .0$	and	$+1\ .0$	13 errors
Between	$+1\ .0$	and	$0\ .0$	26 errors
Between	$0\ .0$	and	$-1\ .0$	26 errors
Between	$-1\ .0$	and	$-2\ .0$	17 errors
Between	$-2\ .0$	and	$-3\ .0$	8 errors
Between	$-3\ .0$	and	$-4\ .0$	2 errors
Total			....	100 errors

Here over one half of the errors lie between $+1''.0$ and $-1''.0$; there are 47 positive and 53 negative errors,

while the largest error is less than 6″.0, thus showing a substantial agreement with the three laws.

Problem 1. Eight measurements of a quantity give the values 186.4, 186.3, 186.2, 186.3, 186.3, 186.2, 185.9, and 186.4, and the most probable value is their arithmetical mean. Compute the eight residuals; find the sum of the positive residuals and the sum of the negative residuals.

SECTION II. THE FUNDAMENTAL PRINCIPLE OF LEAST SQUARES.

The Method of Least Squares sets forth the processes by which the most probable values of observed quantities are derived from the observations. The foundation of the method is the following principle:

> In observations of equal precision the most probable values of observed quantities are those that render the sum of the squares of the residual errors a minimum.

This principle was first enunciated by LEGENDRE in 1805, and has since been universally accepted and used as the basis of the science of the adjustment of observations. The proof of the principle from the theory of mathematical probability requires more space than can be here given, and the plan will be adopted of taking it for granted.* Indeed some writers have regarded the principle as axiomatic.

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* See MERRIMAN'S *Text-book on the Method of Least Squares* (sixth edition, 1892) for two different proofs.

Observations have equal precision when all are made with the same care, or when there is no *a priori* reason to suppose that one is more reliable than another: they are then said to have equal "weight." Weights are numbers expressing the relative practical worth of observations, so that an observation of weight p is worth p times as much as an observation of weight unity. Thus if a line be measured five times with the same care, three measurements giving 950.6 feet and two giving 950.4 feet, then the numbers 3 and 2 are the weights of the observations 950.6 and 950.4. Thus "950.6 with a weight of 3" expresses the same as the number 950.6 written down three times and regarded each time as having a weight of unity; or "950.6 with a weight of 3" might mean that 950.6 is the average of three observations of equal weight.

Let M_1, M_2, M_3, etc., be observations upon quantities whose most probable values are z_1, z_2, and z_3. Then the residual errors are

$$v_1 = z_1 - M_1, \quad v_2 = z_2 - M_2, \quad v_3 = z_3 - M_3, \quad \text{etc.}$$

and by the principle of Least Squares the values to be found for z_1, z_2, and z_3 must be such that

$$v_1^2 + v_2^2 + v_3^2 + \text{etc.} = \text{a minimum.} \quad . \quad .. \quad (2)$$

Now suppose that there are p_1 observations having the value M_1, or that M_1 has the weight p_1; also that M_2 and M_3 have the weights p_2 and p_3. Then there will be p_1 residuals having the value v_1, p_2 having the value v_2, and p_3 having the value v_3. Thus the condition becomes

$$p_1v_1^2 + p_2v_2^2 + p_3v_3^2 + \text{etc.} = \text{a minimum.} \quad . \quad (2)'$$

Hence a more general statement of the principle of Least Squares is:

> In observations of unequal precision the most probable values of the observed quantities are those which render the sum of the weighted squares of the residual errors a minimum.

Here it is seen that the term "weighted square of a residual" means the product of the square of the residual by its weight.

Problem 2. There are taken upon a single quantity the three observations 792.4, 792.6, and 793.1, all of equal weight. Compute the sum of the squares of the residual errors, supposing (*a*) that the most probable value of the quantity is 792.69, (*b*) that it is 792.70, and (*c*) that it is 792.71.

SECTION III. THE PROBABLE ERROR.

The Method of Least Squares comprises two tolerably distinct divisions. The first is the adjustment of observations, or the determination of the most probable values of observed quantities. The second is the investigation of the precision of observations and of the adjusted results. The first is done by the application of the principle of Least Squares given in Section II; the second is done by the determination of the probable error, the rules for which will be given in the sequel.

The following may be stated as a definition of the term "probable error:"

> In any large series of errors the probable error is an error of such a value that the number of errors

less than it is the same as the number greater than it. It is then an even chance that an error taken at random is greater than the probable error, and also an even chance that it is less than the probable error.

To render more definite the conception of probable error let two sets of observations made upon the length of a line be considered. The first set, made with a chain, gives 634.7 feet with a probable error of 0.3 feet. The second set, made with a tape, gives 634.64 with a probable error of 0.06 feet; thus,

$$L_1 = 634.7 \pm 0{\cdot}3 \qquad \text{and} \qquad L_2 = 634.64 \pm 0.06\,;$$

and it is an even chance that 634.7 is within 0.3 of the truth, and also an even chance that 634.64 is within 0.06 of the truth. The probable error thus gives an absolute idea of the accuracy of the results; it also serves as a means of comparing the precision of different observations, for in the above case the precision of the second result is to be taken as much greater than that of the first.

It is a principle of the Method of Least Squares that weights of observations are reciprocally proportional to the squares of their probable errors. Thus, for the above numerical example,

$$p_1 : p_2 :: \frac{1}{0.09} : \frac{1}{0.0036} :: 1 : 25.$$

Hence the second observation has a value about 25 times that of the first when it is to be used in combination with other measurements. Weights and probable errors

are constantly employed in discussing observations by the help of the Method of Least Squares. Weights are usually determined from the number of observations or from knowledge of the manner in which the measurements are made, but probable errors are computed from the observations themselves, and the rules for doing this will be given in subsequent Sections. Weights are relative abstract numbers, but probable errors are absolute concrete quantities.

Problem 3. The probable error of each of the observations given in Problem 2 is 0.24. Find the probable error of the arithmetical mean of the three observations.

Section IV. The Arithmetical Mean.

When observations of equal precision are directly made upon a single quantity it is universally recognized that their arithmetical mean is the most probable value of the quantity. This will now be deduced from the fundamental principle of Least Squares.

Let z be the most probable value of the quantity whose n observed values are $M_1, M_2, \ldots M_n$, all being of equal weight. Then the residual errors are

$$v_1 = z - M_1, \quad v_2 = z - M_2, \quad \ldots \quad v_n = z - M_n;$$

and from the fundamental principle (2),

$$(z - M_1)^2 + (z - M_2)^2 + \ldots + (z - M_n)^2 = \text{a minimum.}$$

To find the value of z which renders this expression a minimum it is to be differentiated and the derivative placed equal to zero, giving

$$2(z - M_1) + 2(z - M_2) \ldots + 2(z - M_n) = 0$$

from which the value of z is

$$z = \frac{M_1 + M_2 + \ldots + M_n}{n};$$

that is, the most probable value of the quantity is the arithmetical mean of the observations.

The rules for finding the probable error of a single observation and of the most probable value will be found demonstrated in the text-book quoted in the foot-note on page 94. Let Σv^2 denote the sum of the squares of all the residuals, or

$$\Sigma v^2 = v_1^2 + v_2^2 + \ldots + v_n^2.$$

Then the probable error of any single observation is

$$r = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}}, \quad \ldots \ldots \quad (4)$$

and the probable error of the arithmetical mean is

$$r_0 = \frac{r}{\sqrt{n}} \quad \ldots \ldots \quad (4)'$$

Thus to compute r and r_0 it is necessary first to find z and the residuals $v_1, v_2, \ldots v_n$.

As an example consider the following twenty-four observations of an angle of the primary triangulation of the United States Coast Survey :

Observations.	v.	v^2.
116° 43′ 44″.45	5.19	26.94
50 .55	− 0.91	0.83
50 .95	− 1.31	1.72
48 .90	0.74	0.55
49 .20	0.44	0.19
48 .85	0.79	0.63
47 .40	2.24	5.02
47 .75	1.89	3.57
51 .05	− 1.41	2.00
47 .85	1.79	3.20
50 .60	− 0.96	0.92
48 .45	1.19	1.42
51 .75	− 2.11	4.45
49 .00	0.64	0.41
52 .35	− 2.71	7.34
51 .30	− 1.66	2.75
51 .05	− 1.41	2.00
51 .70	− 2.06	4.24
49 .05	0.59	0.35
50 .55	− 0.91	0.83
49 .25	0.39	0.15
46 .75	2.89	8.35
49 .25	0.39	0.15
53 .40	− 3.76	14.14
z = 116° 43′ 49″.64		$\Sigma v^2 = 92.15$

The most probable value of the angle is found by adding the observations and dividing their sum by 24. This gives the mean 116° 43′ 49″.64. Subtracting from this the first observation gives 5.19 for the first residual, and the square of this is 26.94. The sum of the squares of all the residuals is 92.15. Then from (4) the probable error of a single observation is

$$r = 0.6745\sqrt{\frac{92.15}{23}} = 1''.35,$$

and from (4)′ the probable error of the mean is

$$r_0 = \frac{1.35}{\sqrt{24}} = 0''.28;$$

hence the final value may be written $116° 43' 49''.64 \pm 0''.28$. Thus the precision of the mean is such that it is an even chance that it differs from the true angle by more or by less than by $0''.28$. The precision of a single observation is such that in a large number half the errors should be greater and half less than $1''.35$. In this example it will be noticed that twelve residuals are greater and twelve less than $1''.35$.

It should be borne in mind that the method of the arithmetical mean only applies to equally good observations on a single quantity, and that it cannot be used when observations are made on several related quantities. For instance, let an angle be measured and found to be $60\frac{1}{2}$ degrees, and again be measured in two parts, one being found to be 40 and the other 20 degrees. The proper adjusted value of the angle is not, as might at first be supposed, the mean of $60\frac{1}{2}$ and 60, which is $60\frac{1}{4}$ degrees, but, as will be seen later, it is $60\frac{1}{3}$ degrees, a result which requires each observation to be corrected the same amount.

Problem 4.—Compute the mean and the probable errors for the first twelve observations in the above example. Also for the last twelve.

Section V. The General Mean.

When observations of unequal precision are directly made upon a single quantity, the method for finding its most probable value is well known to be,

> Multiply each observation by its weight and divide the sum of the products by the sum of the weights.

In order to prove this, let z be the most probable value of the quantity whose observed values are M_1, M_2, ... M_n having the weights p_1, p_2, ... p_n. Then from the general principle of Least Squares given by (2),

$$p_1(z-\mathrm{M}_1)^2+p_2(z-\mathrm{M}_2)^2+\ldots+p_n(z-\mathrm{M}_n)^2=\text{a minimum.}$$

The first derivative of this, placed equal to zero, gives

$$p_1(z-\mathrm{M}_1)+p_2(z-\mathrm{M}_2)+\ldots+p_n(z-\mathrm{M}_n)=0,$$

from which the value of z is

$$z=\frac{p_1\mathrm{M}_1+p_2\mathrm{M}_2+\ldots+p_n\mathrm{M}_n}{p_1+p_2+\ldots+p_n},$$

which agrees with the rule as stated above. The value of z thus found is sometimes called the general mean.

To compute the probable errors the residuals are to be first found. Let Σp denote the sum of all the weights, and Σpv^2 the sum of the weighted squares of the residuals. Then if n denote the number of observations,

$$r=0.6745\sqrt{\frac{\Sigma pv^2}{n-1}}$$

gives the probable error of an observation of the weight unity. Finally,

$$r_0=\frac{r}{\sqrt{\Sigma p}}$$

gives the probable error of the general mean.

As an example, suppose the observations in the example of Section IV to be stated as in the following table, the mean of the first five being 48″.81 with the weight 5, the mean of the next four 48″.76 with the weight 4, and so on. Then the general mean z, of course, agrees with the result before found, and its weight Σp is the sum of the several weights or the number of single measurements. Subtracting each M from z gives the residuals in the column v, and from a table of squares the numbers in the column v^2 are found; multiplying each of these by the corresponding weight gives the quantities in the column pv^2, whose sum is 18.95. Then, since n is 6, the probable error of an observation of weight unity is

M	p	v	v^2	pv^2
116° 43′ 48″.81	5	0.83	0.69	3.45
48 .76	4	0.88	0.77	3.08
49 .53	5	0.11	0.01	0.05
51 .56	3	−1.92	3.69	11.07
50 .38	2	−0.74	0.55	1.10
49 .84	5	−0.20	0.04	0.20
$z = 116° 43′ 49″.64$	$24 = \Sigma p$			$\Sigma v^2 = 18.95$

$$r = 0.6745 \sqrt{\frac{18.95}{5}} = 1''.32,$$

and the probable error of the general mean is

$$r = \frac{1.34}{\sqrt{24}} = 0''.27.$$

These values agree satisfactorily with those found in the previous discussion.

The probable error of any observation is found by dividing its weight into the probable error for an observation of weight unity. Thus for the second observation of the above example the probable error is 0″.66.

Problem 5.—Two observations of an angle are 16° 27′ 33″ and 16° 27′ 36″, with weights 12 and 5. Find the most probable value and its probable error.

Section VI. Observation Equations.

When observations are taken of several related quantities, the measurements are usually made upon functions of those quantities. Thus the sum and difference of two quantities might be observed instead of the

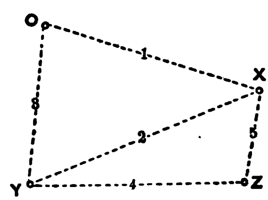

Fig. 11.

quantities themselves. Such observations are generally represented by equations which will be called "observation equations." To illustrate how they arise, let the following practical case be considered. Let O represent a bench-mark, and X, Y, Z, three points whose elevations above O are to be determined. Let five lines of levels be run between those points, giving the results—

Observation 1.	X above O = 10.35 feet.
Observation 2.	Y above X = 7.25 feet.
Observation 3.	Y above O = 17.63 feet.
Observation 4.	Y above Z = 9.10 feet.
Observation 5.	Z below S = 1.94 feet.

Here it will be at once perceived that the measurements are discordant; if observations 1 and 2 are taken as correct, the elevation of X is 10.35 feet, and that of Y is 17.60 feet; if 2 and 3 are correct, then X is 10.38 feet and Y is 17.63 feet; and it will be found impossible to deduce values that will exactly satisfy all the observations. Let the elevations of the points X, Y, and Z above O be denoted by x, y, and z, then the observations furnish the following equations:

$$
\begin{array}{ll}
1. & x = 10.35; \\
2. & g - x = 7.25; \\
3. & y = 17.63; \\
4. & y - z = 9.10; \\
5. & x - z = 1.94.
\end{array}
$$

Each of which is an approximation to the truth. The number of these equations is five, the number of the unknown quantities is three, and hence an exact solution cannot be made. The best that can be done is to find values for x, y, and z which are the most probable, and these will be found in the next Section by the help of the principle of Least Squares.

Observations are called "direct" when made upon the quantity whose value is sought, and "indirect" when made upon functions of the quantities whose value are required. Thus in the above example the first and third observations are direct, and the others are indirect, being made upon differences of elevation instead of upon the elevations themselves. Indirect observations are of frequent occurrence in the operations of precise surveying.

Quantities are said to be "independent" when each can vary without affecting the value of the others; thus

in the above example the elevation of any one station above the bench-mark O is entirely independent of the elevation of the other. Quantities are said to be ".conditioned" when they are so related that a change in one necessarily affects the values of the others; thus if the three angles of a plane triangle be called x, y, and z, it is necessary that $x + y + z = 180°$ and the values to be found for the angles must satisfy this condition. In stating observation equations it will often be found best to select the quantities to be determined in such a way that they shall be independent; thus if the three angles of a triangle are observed to be 62° 20′ 43″, 36° 14′ 06″, and 81° 25′ 08″, let x and y denote the most probable values of the first and second angles, then the observation equations are

$$
\begin{aligned}
x &= 62°\ 20'\ 43'';\\
y &= 36\ \ 14\ \ 06;\\
180° - x - y &= 81\ \ 25\ \ 08;
\end{aligned}
$$

the last of which may be written

$$x + y = 98°\ 34'\ 52'',$$

and here x and y are independent quantities. In this book the quantities to be determined from observations are always, unless otherwise stated, to be so selected as to be independent. This case will be briefly called that of "independent observations"

As a second example of the statement of observation equations take the following values of the angles measured at North Base, Keweenaw Point, on the United States Lake Survey:

$$
\begin{array}{rcrrl}
\text{CNM} & = & 55^\circ & 57' & 58''.68\,; \\
\text{MNQ} & = & 48 & 49 & 13\,.64\,; \\
\text{CNQ} & = & 104 & 47 & 12\,.66\,; \\
\text{QNS} & = & 54 & 38 & 15\,.53\,; \\
\text{MNS} & = & 103 & 27 & 28\,.99.
\end{array}
$$

The object of these observations is to find the values of the four angles around the point N; but if x, y, z, and w represent these angles, then $x + y + z + w = 360^\circ$ and the quantities are conditioned. To make the quantities independent only three unknowns should be taken; thus let CNM $= x$, MNQ $= y$, and QNS $= z$, then the observation equations are

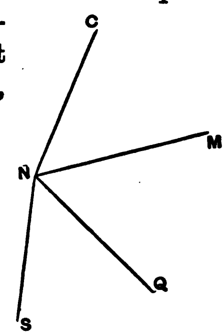

Fig. 12.

$$
\begin{array}{rcrrl}
x & = & 55^\circ & 57' & 58''.68\,; \\
y & = & 48 & 49 & 13\,.64\,; \\
x + y & = & 104 & 47 & 12\,.66\,; \\
z & = & 54 & 38 & 15\,.53\,; \\
y + z & = & 103 & 27 & 28\,.99\,;
\end{array}
$$

and in the next Section it will be shown how the most probable values of x, y, and z are to be found.

Thus, in general, observations upon several quantities lead to observation equations whose number is greater than that of the unknown quantities, and no system of values can be found that will exactly satisfy the observation equations. They may, however, be approximately satisfied by many systems of values; and the problem is to determine that system which is the most probable and hence the best.

Problem 6. State observation equations for the above

example, taking CNM $= x$, CNQ $= y$, and CNS $= z$. Also taking SNQ $= s$, SNM $= t$, and MNC $= u$.

Section VII. Independent Observations.

Let $M_1, M_2, \dots M_n$ be n observations of equal weight made upon functions of the unknown quantities x, y, z, etc. Let the observations give rise to the following observation equations,

$$
\begin{aligned}
a_1x + b_1y + c_1z + \dots &= M_1\,; \\
a_2x + b_2y + c_2z + \dots &= M_2\,; \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_nx + b_ny + c_nz + \dots &= M_n\,;
\end{aligned}
$$

in which $a_1, a_2, \dots a_n, b_1, b_2, \dots b_n$, etc., are known coefficients of the unknown quantities. The most probable values of x, y, z, etc., when found and inserted in the equations will not exactly satisfy them, but leave small residual errors, $v_1, v_2, \dots v_n$; thus strictly

$$
\begin{aligned}
a_1x + b_1y + c_1z + \dots - M_1 &= v_1; \\
a_2x + b_2y + c_2z + \dots - M_2 &= v_2; \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_nx + b_ny + c_nz + \dots - M_n &= v_n;
\end{aligned}
$$

and by the principle of Least Squares the sum of the squares of these residuals must be a minimum in order to give the most probable values of x, y, and z.

In order to find the condition for the most probable value of x let the terms independent of x in the equations

be denoted by N_1, N_2, . . . N_n; then they may be written

$$\begin{aligned} a_1x + N_1 &= v_1 \\ a_2x + N_2 &= v_2; \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ a_nx + N_n &= v_n. \end{aligned}$$

Squaring both terms of these equations, and adding, gives

$$(a_1x + N_1)^2 + (a_2x + N_2)^2 + \ldots + (a_nx + N_n)^2 = \Sigma v^2,$$

and this is to be made a minimum to give the most probable value of x. Differentiating it with respect to x, placing the first derivative equal to zero and dividing by 2, gives

$$a_1(a_1x + N_1) + a_2(a_2x + N_2) + \ldots + a_n(a_nx + N_n) = 0,$$

and this is the condition for the most probable value of x. In like manner a similar condition may be stated for each of the other unknown quantities. The conditions thus stated are called "normal equations," and their solution will furnish the most probable values of the required quantities:

The following is hence the rule for the adjustment of observations of equal weight upon several independent quantities,

> For each of the unknown quantities form a normal equation by multiplying each observation equation by the coefficient of that unknown quantity in that equation and adding the results. Then the solution of these normal equations will furnish the most probable values of the unknown quantities.

In forming the normal equations it should be particularly noticed that the signs of coefficients are to be observed in performing the multiplications, and also that when the unknown quantity under consideration does not occur in an observation equation its coefficient is 0.

As an example the five observation equations at the beginning of the last Section will be taken. They may be written thus:

$$\begin{array}{llcl}
1. & x & = & 10.35; \\
2. & -x+y & = & 7.25; \\
3. & \quad\; y & = & 17.63; \\
4. & \quad\; y-z & = & 9.10; \\
5. & x \quad\; -z & = & 1.94.
\end{array}$$

Now to form the normal equation for x, the first equation is to be multiplied by 1, the second by -1, and the fifth by 1; and adding these,

$$3x - y - z = 5.03.$$

In like manner to find the normal equation for y, the second equation is multiplied by 1, the third by 1, and the fourth by 1, whence

$$-x + 3y - z = 33.98.$$

Lastly, to find the normal equation for z, the fourth equation is multiplied by -1 and the fifth by -1, and adding,

$$-x - y + 2z = -11.04.$$

These three normal equations contain three unknown quantities, and their solution gives

$$x = 10.372, \qquad y = 17.61, \qquad z = 8.47 \text{ feet},$$

which are the most probable values of the three elevations.

As a second example the three observation equations near the middle of the last Section are

$$\begin{aligned} x \qquad &= 62^\circ\ 20'\ 43''; \\ y &= 36\ \ 14\ \ 06; \\ x + y &= 98\ \ 34\ \ 52. \end{aligned}$$

Applying the rule, the two normal equations are

$$\begin{aligned} 2x + y &= 160\ \ 55\ \ 35; \\ x + 2y &= 134\ \ 48\ \ 58; \end{aligned}$$

and the solution of these gives

$$x = 62^\circ\ 20'\ 44'', \qquad y = 36^\circ\ 14'\ 07'',$$

whence the third angle of the triangle is 180 degrees minus the sum of these, or $81^\circ\ 25'\ 09''$. By comparing these with the observed values it will be seen that each observation is corrected by the same amount; this is because the observations are of equal weight and each angle is similarly related to the other two.

Problem 7. Form and solve the normal equations for the angle observations given near the end of Section VI.

Section 8. Independent Observations with Weights.

The two preceding Sections give the method of adjusting observations of equal weight upon several independent quantities; now is to be investigated the case of observations of different weights upon such quantities. Let $P_1, P_2, \ldots P_n$ be the weights of the observations $M_1, M_2, \ldots M_n$, so that the observation equations are

$$a_1x + b_1y + c_1z + \ldots = M_1, \quad \text{with weight } p_1;$$
$$a_2x + b_2y + c_2z + \ldots = M_2, \quad \text{with weight } p_2;$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_nx + b_ny + c_nz + \ldots = M_n, \quad \text{with weight } p_n.$$

Now if the first equation were written p_1 times, the second p_2 times, etc., all the equations would have the same weight and the rule of the last Section would apply. That is, if each of the above equations be multiplied by the coefficient of x in that equation, and also by its weight, the sum will be the condition for the most probable value of x; and in like manner is found the condition for the most probable value of each of the other unknowns. These conditions are the normal equations.

The following is hence the rule for the adjustment of observations of unequal weight upon several independent quantities;

For each of the unknown quantities form a normal equation by multiplying each observation equation by the coefficient of that unknown quantity in that equation, and also by its weight, and add-

ing the results. The solution of these normal equations will furnish the most probable values of the unknown quantities.

In applying this rule the same precautions are to be observed regarding signs of the coefficients as before stated.

An algebraic expression of the normal equations can be made by introducing the following abbreviations:

$$\begin{aligned}
\Sigma pa^2 &= p_1a_1^2 + p_2a_2^2 + \ldots + p_na_n^2;\\
\Sigma pab &= p_1a_1b_1 + p_2a_2b_2 + \ldots + p_na_nb_n;\\
\Sigma pa\mathrm{M} &= p_1a_1\mathrm{M}_1 + p_2a_2\mathrm{M}_2 + \ldots + p_na_n\mathrm{M}_m;\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

and then the normal equations can be written

$$\begin{aligned}
\Sigma pa^2 \,.\, x + \Sigma pab \,.\, y + \Sigma pac \,.\, z + \text{etc.} &= \Sigma pa\mathrm{M};\\
\Sigma pab \,.\, x + \Sigma pb^2 \,.\, y + \Sigma pbc \,.\, z + \text{etc.} &= \Sigma pb\mathrm{M};\\
\Sigma pac \,.\, x + \Sigma pbc \,.\, y + \Sigma pc^2 \,.\, z + \text{etc.} &= \Sigma pc\mathrm{M};\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&
\end{aligned}$$

Here it will be seen that the coefficients of the unknown quantities in the first vertical column are the same as those in the first horizontal line, those in the second column the same as those in the second line, and so on. This is a characteristic of normal equations and serves as a check when they are deduced by direct application of the rule. If the observations are of equal weight, p is to be made unity throughout, and the method reduces to that before given.

As a numerical illustration let five observations produce the five observation equations—

$$
\begin{array}{llll}
1. & +x \qquad\qquad = & 0, & \text{with weight } 3; \\
2. & \qquad +y \qquad = & 0, & \text{with weight } 19; \\
3. & \qquad\qquad +z = & 0, & \text{with weight } 13; \\
4. & +x+y \qquad = & +0.34, & \text{with weight } 17; \\
5. & \qquad +y+z = & -0.18, & \text{with weight } 6.
\end{array}
$$

From these the normal equations, formed either by the rule or by help of the algorithm, are

$$
\begin{aligned}
20x + 17y \qquad\quad &= +5.78; \\
17x + 42y + \ 6z &= +4.70; \\
6y + 19z &= -1.08;
\end{aligned}
$$

whose solution furnishes the results

$$x = +0.285, \qquad y = +0.005, \qquad z = -0.059,$$

which are the most probable values of the required quantities.

Problem 8. In a plane triangle six observations give A $= 42° 17' 35''$, three observations give B $= 56° 40' 09''$, and two observations give C $= 81° 02' 10''$. Compute the adjusted values of the angles.

Section IX. Probable Errors of Independent Observations.

It is sometimes required to find the probable errors of the observed quantities $M_1, M_2, \ldots M_n$ and the probable errors of the most probable values of x, y, z, etc.

These may be found by first deducing the probable error of an observation of the weight unity and then dividing this by the weights $p_1, p_2, \ldots p_n$ and p_x, p_y, p_z, etc. If n is the number of observations, q the number of unknown quantities, and Σpv^2 the sum of the weighted squares of the residuals, then

$$r = 0.6\,745 \sqrt{\frac{\Sigma pv^2}{n-q}} \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

is the formula for the probable error of an observation of the weight unity, and

$$r_1 = \frac{r}{\sqrt{p_1}}, \qquad r_x = \frac{r}{\sqrt{p^x}}$$

are the probable errors of M_1 and of x respectively.

The weights $p_1, p_2, \ldots p_n$ are known, but the weights p_x, p_y, etc., are to be derived by preserving the absolute terms of the normal equations in literal form during the solution. Then the weight of any unknown quantity is the reciprocal of the coefficient of the absolute term which belongs to the normal equation for that unknown quantity. For instance, take the normal equations

$$\begin{aligned} 3x - \ y - z &= A; \\ -x + 3y - z &= B; \\ -x - y + 2z &= C. \end{aligned}$$

The solution of these by any method gives

$$\begin{aligned} x &= \tfrac{5}{8}A + \tfrac{3}{8}B + \tfrac{1}{2}C; \\ y &= \tfrac{3}{8}A + \tfrac{5}{8}B + \tfrac{1}{2}C; \\ z &= \tfrac{1}{2}A + \tfrac{1}{2}B + C. \end{aligned}$$

Hence the weight of x is $\frac{8}{5}$, that of y is $\frac{8}{5}$, and that of z is 1. If it be only desired to find the weight of x, the terms B and C need not be retained in the computation; if only to find the weight of z, the terms A and B can be omitted.

As a numerical example the observation equations given at the beginning of Section VI may again be considered. These may be written, if x, y, and z denote the most probable elevations,

$$\begin{aligned} x - 10.35 &= v_1; \\ y - x - 7.25 &= v_2; \\ y - 17.63 &= v_3; \\ y - z - 9.10 &= v_3; \\ x - z - 1.94 &= v_4; \end{aligned}$$

in which v_1, v_2, etc., are the residual errors. Now in Section VII the most probable values were derived,

$$x = 10.37, \qquad y = 17.61, \qquad \text{and} \qquad z = 8.47 \text{ feet},$$

and substituting these, the residuals are found to be

$$v_1 = +0.02, \quad v_2 = -0.01, \quad v_3 = -0.02, \quad v_4 = +0.04, \quad v_5 = -0.04.$$

Now, as the weights are equal, Σpv^2 becomes Σv^2, and its value is $\Sigma v^2 = 0.0041$. Then, since n is 5 and q is 3,

$$r = 0.6745\sqrt{\frac{0.0041}{5-3}} = 0.031 \text{ feet},$$

which is the probable error of a single observation.

By the method above explained it will be found that the weight of x is 1.8, whence its probable error is

$$r_x = \frac{0.031}{\sqrt{1.8}} = 0.023 \text{ feet};$$

and in a similar manner the probable errors of y and z are 0.023 feet and 0.031 feet. The final adjusted values may then be written

$$x = 10.37 \pm 0.02, \quad y = 17.61 \pm 0.02, \quad z = 8.47 \pm 0.03.$$

Problem 9. Four measurements give the observation equations

$$\begin{aligned} +x \qquad\qquad &= 12.27, \quad \text{with weight } 2; \\ -x + y \qquad &= \ \ 1.04, \quad \text{with weight } 2; \\ -y + z &= \ \ 3.30, \quad \text{with weight } 1; \\ z &= 16.67, \quad \text{with weight } 1. \end{aligned}$$

Find the most probable values of x, y, and z, their weights and their probable errors.

Section X. Conditioned Observations.

Although by the proper selection of the unknown quantities it is generally possible to state observation equations so that the quantities will be independent (Section VIII), yet cases sometimes arise in which this method is not convenient. In such cases the quantities x, y, z, etc., are connected by theoretical conditional equations which must be exactly satisfied, while the most probable values to be found for them must render

the sum of the weighted squares of the residual errors a minimum. The following method, known as the "method of correlatives," is one extensively used in the adjustment of geodetic triangulations:

Consider q unknown quantities connected by the n' theoretical conditions

$$\left.\begin{aligned} \alpha_0 + \alpha_1 x + \alpha_2 y + \ldots &= 0; \\ \beta_0 + \beta_1 x + \beta_2 y + \ldots &= 0; \\ \gamma_0 + \gamma_1 x + \gamma_2 y + \ldots &= 0; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

Let M_1, M_2, etc., be approximate values of x, y, etc., found from the observations, and let p_1, p_2, etc., be their weights. If M_1, M_2, etc., be inserted for x, y, etc., in the conditional equations, they will not reduce to zero, but leave small discrepancies d_1, d_2, etc.; thus,

$$\begin{aligned} \alpha_0 + \alpha_1 M_1 + \alpha_2 M_2 + \ldots &= d_1; \\ \beta_0 + \beta_1 M_1 + \beta_2 M_2 + \ldots &= d_2; \\ \gamma_0 + \gamma_1 M_1 + \gamma_2 M_2 + \ldots &= d_3; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Now let K_1, K_2, etc., be certain unknown quantities called "correlatives," and let the following correlative equations be formed:

$$\begin{aligned} \Sigma\frac{\alpha^2}{p} \cdot K_1 + \Sigma\frac{\alpha\beta}{p} K_2 + \Sigma\frac{\alpha\gamma}{p} \cdot K_3 + \ldots + d_1 &= 0; \\ \Sigma\frac{\alpha\beta}{p} \cdot K_1 + \Sigma\frac{\beta^2}{p} \cdot K_2 + \Sigma\frac{\beta\gamma}{p} \cdot K_3 + \ldots + d_2 &= 0; \\ \Sigma\frac{\alpha\gamma}{p} \cdot K_1 + \Sigma\frac{\beta\gamma}{p} \cdot K_2 + \Sigma\frac{\gamma^2}{p} \cdot K_3 + \ldots + d_3 &= 0; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

in which the usual notation for sums is followed, namely,

$$\Sigma\frac{\alpha^2}{p} = \frac{\alpha_1^2}{p_1} + \frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3} + \dots;$$

$$\Sigma\frac{\alpha\beta}{p} = \frac{\alpha_1\beta_1}{p_1} + \frac{\alpha_2\beta_2}{p_2} + \frac{\alpha_3\beta_3}{p_3} + \dots.$$

The values of K_1, K_2, etc., having been deduced by the solution of these normals, let the residuals be computed by the formulas

$$v_1 = \frac{1}{p_1}(\alpha_1 K_1 + \beta_1 K_2 + \gamma_1 K_3 + \dots);$$

$$v_2 = \frac{1}{p}(\alpha_2 K_1 + \beta_2 K_2 + \gamma_2 K_3 + \dots).$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Then the final most probable values of x, y, etc., are

$$x = M_1 + v_1, \qquad y = M_2 + v_2, \qquad \text{etc.,}$$

and these will exactly satisfy the theoretical conditional equations.

As an illustration of the method, let there be five observations on five quantities :

1. $x = 0,$ with weight 3;
2. $y = 0,$ with weight 19;
3. $z = 0,$ with weight 13;
4. $w = +0.34,$ with weight 17;
5. $u = 0.18,$ with weight 6;

which are subject to the two rigorous theoretical conditions

$$x + y \quad - w \quad = 0;$$
$$y + z \quad - u = 0.$$

By substituting in these the observed values there is found

$$d_1 = -0.34, \qquad d_2 = +0.18.$$

Now by comparison with the above algorithm it is seen that

$$\alpha_0 = 0,\ \ \alpha_1 = 1,\ \ \alpha_2 = 1,\ \ \alpha_3 = 0,\ \ \alpha_4 = -1,\ \ \alpha_5 = 0;$$
$$\beta_0 = 0,\ \ \beta_1 = 0,\ \ \beta_2 = 1,\ \ \beta_3 = 1,\ \ \beta_4 = 0,\ \ \beta_5 = -1;$$

and then

$$\Sigma\frac{\alpha^2}{p} = \frac{1}{3} + \frac{1}{19} + 0 + \frac{1}{17} + 0 = 0.4\,448;$$

$$\Sigma\frac{\alpha\beta}{p} = 0 + \frac{1}{19} + 0 + 0 \ + 0 = 0.0\,526;$$

$$\Sigma\frac{\beta^2}{p} = 0 + \frac{1}{19} + \frac{1}{13} + 0 + \frac{1}{6} = 0.2\,962.$$

The two correlative equations now are

$$0.4\,448K_1 + 0.0\,526K_2 = +0.34;$$
$$0.0\,526K_1 + 0.2\,962K_2 = -0.18;$$

whose solution gives the values

$$K_1 = +0.855, \qquad K_2 = -0.759.$$

The residual corrections then are

$$\begin{aligned}
v_1 &= \tfrac{1}{3}(K_1 + 0) &&= + 0.285\,; \\
v_2 &= \tfrac{1}{19}(K_1 + K_2) &&= + 0.005\,; \\
v_3 &= \tfrac{1}{13}(0 + K_2) &&= - 0.059\,; \\
v_4 &= \tfrac{1}{17}(- K_1 + 0) &&= - 0.050\,; \\
v_5 &= \tfrac{1}{6}(0 - K_2) &&= + 0.126\,;
\end{aligned}$$

from which the final adjusted values are

$$\begin{aligned}
x &= 0 + 0.285 = + 0.285\,; \\
y &= 0 + 0.005 = + 0.005\,; \\
z &= 0 - 0.059 = - 0.059\,; \\
w &= + 0.34 - 0.050 = + 0.290\,; \\
u &= - 0.18 + 0.126 = - 0.054.
\end{aligned}$$

And it will be seen that these exactly satisfy the two rigorous conditional equations.

The probable error of an observation of the weight unity is given by the formula

$$r = 0.6\,745\sqrt{\frac{\Sigma pv^2}{n - q + n'}},$$

in which n is the number of observation equations, q the number of unknown quantities, and n' the number of conditional equations. For the above example the residuals are already found; squaring them, multiplying each by its weight, and adding, gives $\Sigma pv^2 = 0.428$, whence

$$r = 0.6\,745\,\sqrt{0.214} = 0.309$$

for the probable error of an observation of weight unity.

Problem 10. The three observation equations $x = 2$, $y = 5$, $z = 10$ have the respective weights 4, 3, and 2, and are connected by the rigorous conditional equation $x + y + z = 16$. Compute the most probable values of x, y, and z.

Section XI. Laws of Propagation of Error.

The determination of the precision of quantities which are computed from other observed quantities is now to be discussed. For instance, the area of a field is computed from its sides and angles. When the most probable values of these have been found by measurement, the most probable value of the area is computed by the rules of geometry, and the precision of that area will depend upon the precision of the observed quantities.

Let x_1 and x_2 be two independently measured quantities whose probable errors are r_1 and r_2, and whose sum is X. Then R, the probable error of X, is

$$R = \sqrt{r_1^2 + r_2^2}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The same formula also applies for the probable error of the difference of two observed quantities. In like manner, if X be the algebraic sum of several independent quantities; namely, if

$$X = x_1 \pm x_2 \pm x_3 \pm \ldots \pm x_n,$$

then the probable error of X is given by the relation

$$R^2 = r_1^2 + r_2^2 + r_3^2 + \ldots + r^{n2}. \quad . \quad . \quad (11)'$$

This formula is very important in the discussion of linear measurements.

If x be an observed quantity whose probable error is r, then the probable error of ax is ar. Thus, if the diameter of a circle is observed to be 42 feet 2 inches $\pm$ 0.5 inches, the circumference is 132.47 feet $\pm$ 0.13 feet.

Let x_1 and x_2 have the probable errors r_1 and r_2; let $X = x_1x_2$; then the probable error of X is given by

$$R^2 = x_1^2r_2^2 + x_2^2r_1^2.$$

For example, if the sides of a rectangular field be, in feet, 50.00 $\pm$ 0.01 and 200.00 $\pm$ 0.02, the area, in square feet, is 10 000 $\pm$ 2.24.

Lastly, let X be any function of the independently observed quantities x, y, z, etc., and let it be required to find the probable error of X from the probable errors r_1, r_2, etc., of the observed quantities. If the observations be conducted with precision, so that the probable errors are small, this is given by the formula

$$R^2 = \left(\frac{dX}{dx}\right)^2 r_1^2 + \left(\frac{dX}{dy}\right)^2 r_2^2 + \left(\frac{dX}{dz}\right)^2 r_3^2 + \ldots$$

For example, let x be the observed value of the diameter of a circle, and r_1 its probable error; then $X = \frac{1}{4}\pi x^2$, and $dX = \frac{1}{2}\pi x \,.\, dx$, whence $R = \frac{1}{2}\pi x \,.\, r_1$. Thus if x is observed to be 42 feet 2 inches $\pm$ 0.5 inches, the area is 1 396.46 square feet $\pm$ 2.76 square feet.

The formulas for probable errors stated in these pages will be found deduced in text-books on the Method of

Least Squares. By their help the precision of observations is determined, so that the accuracy of results can be judged without knowledge of the method by which the measurements were made. The processes of the Method of Least Squares for the adjustment and comparison of observations are now universally used in astronomy, physics, geodesy, and wherever precise measurements are made. It should therefore be studied as an introduction to Geodesy, and there is little doubt but that in future years all works on Surveying will more or less treat of its principles and processes.

Problem 11. If the probable error of A is 10 seconds, find the maximum probable error of $\sin A + \cos A$, and the value of A when it occurs.

Section XII. Answers to Problems; and Notes.

Problem 1. The sum of the positive residuals is 0.45, and that of the negative ones is also 0.45. It is a property of the arithmetical mean that these two sums are always equal.

Problem 2. 0.2603, 0.2600, and 0.2603.

Problem 3. The arithmetical mean has the probable error 0.14.

Problem 4. $116° 43' 48''.83 \pm 0''.35$ and $116° 43' 50''.45 \pm 0''.36$.

Problem 5. $z = 16° 27' 33''.9 \pm 0''.9$.

Problem 7. In order to avoid large numbers in the computations, it is best to take x_1, y_1, and z_1 as corrections to the measured values of x, y, and z; so that the most probable values of the latter are

$$x = 55^\circ\ 57'\ 58''.68 + x_1;$$
$$y = 48\ \ 49\ \ 13\ .64 + y_1;$$
$$z = 58\ \ 38\ \ 15\ .53 + z_1.$$

If, now, these be substituted in the observation equations on page 107 they reduce to $x_1 = 0$, $y_1 = 0$, $x_1 + y_1 = +0''.34$, $z_1 = 0$, $x_1 + z_1 = -0''.18$. From these the normal equations in x_1, y_1, and z_1 are formed, whose solution gives the most probable values of the corrections. Lastly, the adjusted angles are CNM $= 55^\circ\ 57'\ 58''.83$, CNQ $= 104^\circ\ 47'\ 12''.51$, MNQ $= 48^\circ\ 49'\ 13''.68$, QNS $= 54^\circ\ 38'\ 15''.42$, and MNS $= 103^\circ\ 27'\ 29''.10$.

Problem 8. $A = 42^\circ\ 17'\ 36''$, $B = 36^\circ\ 40'\ 11''$, $C = 81^\circ\ 02'\ 13''$.

Problem 9. $x = 12.28 \pm 0.015$, $y = 13.33 \pm 0.019$, $z = 16.65 \pm 0.019$.

Problem 10. $x = 1\frac{10}{13}$, $y = 4\frac{9}{13}$, and $z = 9\frac{7}{13}$.

Problem 11. The maximum probable error of the function is 0.000 096 963 and it occurs when A is 135 degrees.

Page 52. The book mentioned in the foot-note has been out of print since 1883, but another one, covering the same ground, was issued in 1884 under the title *Text-Book on the Method of Least Squares.*

Pages 52 and 53. The deduction of the empirical formula connecting s and l is here made by supposing that l is free from errors of observation, because plainly no slight variations in the seconds of latitude could produce any sensible change in the length of the pendulum. If two quantities, x and y, are connected by the relation $y = ax + b$, and both x and y are observed, the process here described does not give the most probable values of a and b. The method to be followed may be seen in the *Bulletin of the New York Mathematical Society*, for November 1891, or in the sixth edition of the author's larger work on the method of Least Squares.

Section IV. The fundamental principle of making the sum of the squares of the residuals a minimum is, in strictness, only correct when we have no *a priori* knowledge of the observations. The arithmetical mean of two observations must be the most probable value of the measured quantity, but if there be three observations the mean cannot, in general, be the most probable quantity when the observed values are known. For instance, let three measurements give $M_1 = 50$, $M_2 = 90$, and $M_3 = 100$. The arithmetical mean is 80, and this is derived by supposing the three observations to have equal weight. But as in this case M_1 differs so much from M_2 and M_3, the mind refuses to allow it equal weight, and demands that the probable value should be higher than the arithmetical mean. Although this has long been recognized, no satisfactory method of procedure has been developed to replace that of the arithmetical mean. The following method, though awkward in computation, is suggested as an approximation to a strict theoretical mean. Let the weight of M_1 be the reciprocal of $(M_2 - M_1)^2 + (M_3 - M_1)^2$, the weight of M_2

be the reciprocal of $(M_3 - M_2)^2 + (M_1 - M_2)^2$, and the weight of M_3 be the reciprocal of $(M_1 - M_3)^2 + (M_2 - M_3)^2$; then find the general mean of the weighted observations. For the numerical example in hand this process gives $\frac{1}{41}$, $\frac{1}{17}$, and $\frac{1}{26}$ as the respective weights of 50, 90, and 100, whence by Section V the mean is 85.1.

Two excellent bibliographies of geodetic literature were published in 1889, one by Professor J. Howard Gore in the *U. S. Coast and Geodetic Survey Report* for 1887, and one by Dr. O. Boersch for the International Geodetic Association. The former of these, at least, should be consulted by every student. It is arranged both by authors and by subjects, and in it will be found the titles of numerous books and papers on the Figure of the Earth and on the Method of Least Squares.

THE FIELD WORK

OF

TRIANGULATION.

THE

FIELD WORK OF TRIANGULATION.

ARTICLE 1. GEOGRAPHICAL POSITIONS.

THE latitude and longitude of a station determine its position on the earth's spheroid. In addition to these it is always important to know the distance and direction from the station to each of the neighboring ones, in order to be able to extend the triangulation or use it for local surveys. The complete statement of a geographical position, hence, includes latitude, longitude, azimuth, and distance, the last being reduced to the mean ocean level.

The determination of the astronomical latitude of a station is generally effected by measuring with a zenith telescope the differences of the zenith distances of several pairs of stars. The longitude of a station is found by comparing by means of the electric telegraph the local time of the meridian of the station with that of some astronomical observatory whose longitude is known. The azimuth of one of the lines of the triangulation, or the angle which it makes with the true meridian at one of the stations, is determined by observations on circumpolar stars. For a full account of the astronomical methods of determining latitude, longitude, and azimuth, reference is made to the papers in the U. S. Coast

and Geodetic Survey's *Report* for 1880, pp. 201–286, and also to DOOLITTLE's *Practical Astronomy*, New York, 1885.

The azimuth of a line is an angle which denotes the direction which that line makes with the true meridian. This angle is always measured at one end of the line with respect to a true meridian passing through that end. On the Coast and Geodetic Survey azimuths are reckoned from the south around through west, north, and east,

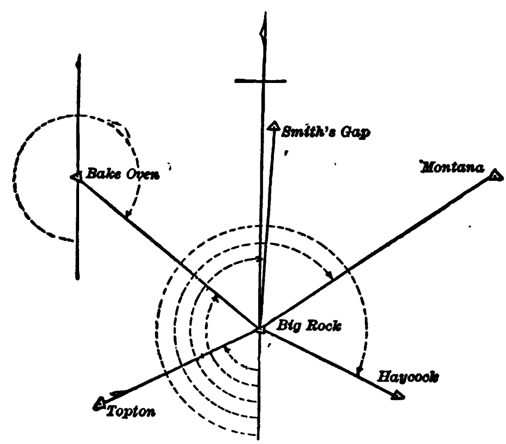

Fig. 13.

from 0° to 360°, so that true south has the azimuth 0°, west 90°, north 180°, and east 270°. Hence the true bearing of a line is at once known when its azimuth has been determined, azimuth and true bearing being, indeed, but different formal statements of the same thing. For example, at the station Big Rock the following are the azimuths and true bearings to the five stations shown in Fig. 13. It will be seen that the azimuths likewise give all the angles between the stations at the point Big

Rock. For instance, the angle Bake Oven–Big Rock–Topton is 64° 18′ 59″.01, the difference of the azimuths Big Rock–Bake Oven and Big Rock–Topton.

At Big Rock.	Azimuth.	True Bearing.
To Topton	64° 54′ 36″.81	S. 64° 54′ 36″.81 W.
To Bake Oven	129 13 35 .82	N. 50 46 24 .18 W.
To Smith's Gap	182 33 18 .74	N. 2 33 18 .74 E.
To Montana	235 05 46 .52	N. 55 05 46 .52 E.
To Haycock	294 32 22 .86	S. 65 27 37 .14 E.

A triangulation may be regarded as a cheap method of determining geographical positions. The astronomical observations for latitude, longitude, and azimuth being first made at one of the stations and a base line being also measured, the field operations and computations of triangulation enable latitude, longitude, azimuth, and distances to be computed for each of the other stations. Thus is laid out a framework upon which topographical surveys may be based, and at each station obtain checks as to its accuracy.

On account of the spheroidal shape of the earth the meridians are not parallel, but converge toward the poles. If the surface of the earth were plane, the azimuth of the direction Bake Oven to Big Rock would be exactly 180° greater than that of the direction Big Rock to Bake Oven, but actually the azimuths are,

Big Rock to Bake Oven,	129° 13′ 35″.82
Bake Oven to Big Rock,	309 02 01 .43
Difference,	179 48 25 .61

So that the convergence of the meridians in this case causes the true bearings of the line to differ by 11′ 34″.39 when taken at opposite ends.

ARTICLE 2. SUBSIDIARY BASE LINES.

Primary base lines are usually several miles in length, and are measured with great precision by special apparatus (see page 39). Concerning these much has been written and in the bibliographies mentioned on page 127 the titles of numerous articles on base apparatus, methods of measurement, and degrees of precision, may be found. See especially *Primary Triangulation of the U. S. Lake Survey*, Washington, 1882.

The following description of the measurement of a subsidiary base with a long steel tape is abridged from a paper on precise triangulation by Prof. H. S. JACOBY, published in the *Lehigh Quarterly* for January, 1891:

The base lines were measured by means of a standard steel tape 400 feet long, manufactured by Heller & Brightly about 1883. Each base line was divided into as many parts as were needed to make each one shorter than the tape, stout plugs being set at these points of division and each point marked by the notch in the head of a small screw placed in the top of the plug. The elevations of these plugs were carefully determined, as well as those of monuments or stations marking the extremities of the line. Each of these main divisions was subdivided into equal parts by light stakes set in line and on the grade between the end plugs, the distance between these stakes being about fifty feet or less. Two nails were placed on the top of the stakes to keep the tape in position. When these preliminaries were arranged for the entire line the actual measurement was begun by suspending the tape over the plugs and stakes as supports, these being high enough for the tape to clear the ground between the stakes when under tension. A

spring balance was attached to the tape at each end of the division to be measured, the pull on the same being recorded. The tape was so held by the balances that at one end a ten-foot mark coincided with the mark on the plug, and at the other end the distance from the nearest foot mark to the mark on the plug was measured by a separate scale to the nearest thousandths of a foot. The temperature of the atmosphere was also taken at the same time and recorded. Six measurements were generally made in succession, two with a pull of 16, two with 18, and two with 20 pounds.

The field notes of one measurement of a short base line EG, about 922 feet long, will illustrate the method of operation. There were three divisions, designated as I, II, and III, the first having six and the others seven

FIELD NOTES. Base Line EG.

Divisions.	No. of Subdivisions.	Diff. in Elevation of Ends	Temperature.	Pull.	Observed Distance.	Date, and Remarks.
		feet	°	lbs.	feet	
III	7	2.813	51	16	309.865	
			50.5	18	309.857	
			50.5	20	309.842	Oct. 3, 1888,
			50	16	309.870	P.M.
			50	18	309.857	
			49.5	20	309.845	
II	7	5.618	48	16	332.736	
			47.5	18	332.727	
			47.5	20	332.712	
			47	16	332.740	
			47	18	332.726	
			47	20	332.715	
I	6	7.924	47	16	279.850	Cloudy and windy.
			47	18	279.843	
			47	20	279.832	
			48	16	279.848	
			48.5	18	279.840	
			48	20	279.837	

subdivisions. The observed length of each division was next corrected for temperature, sag, pull, and grade by using the following formulas: *

$$\text{Correction for temperature} = C_t = +c(T - T_0)l;$$

$$\text{Correction for sag} \qquad = C_s = -\frac{w^2}{24P^2} \cdot \frac{l^3}{n^2};$$

$$\text{Correction for pull} \qquad = C_p = +\frac{(P - P_0)l}{AE};$$

$$\text{Correction for grade} \qquad = C_g = -(l - \sqrt{l^2 - h^2}),$$

$$\text{or } C_g = -\frac{h^2}{2l};$$

in which the letters have the meanings:

c = coefficient of expansion of tape for one degree Fahrenheit.
T = observed temperature.
T_0 = temperature at which the tape is a standard under a given pull.
l = observed length of the division.
n = number of subdivisions in the division.
w = weight per linear foot of the tape.
P = pull on tape at time of reading.
P_0 = pull at which the tape is a standard.
A = area of cross-section of tape.
E = coefficient of elasticity of the tape.
h = difference of elevation between the ends of the division.

The steel tape used was stated by the makers to be standard at 56 degrees Fahrenheit, with a tension of 16

* See JOHNSON'S *Theory and Practice of Surveying*, page 457.

pounds and no sag; hence $T_0 = 56°$ and $P_0 = 16$ pounds. By a series of experiments c had been determined to be 0.00000703, the weight w to be 0.0066 pounds, A to be 0.00199 square inches, and E had been found to be 28 200 000 pounds per square inch. Using these constants the corrections for each observed distance were computed from the formulas and arranged as below.

T	P	Observed Distance.	C_t	C_s	C_p	Corrected Distance.	Div. III.
51.	16	309.865	−.0109	−.0043	0	309.8498	$n = 7.$
50.5	18	309.857	−.0120	−.0034	+.0110	309.8526	
50.5	20	309.842	−.0120	−.0028	+.0221	309.8493	$h = 2.813$ ft.
50.	16	309.870	−.0131	−.0043	0	309.8526	
50.	18	309.857	−.0131	−.0034	+.0110	309.8515	$C_g = -0.0128$ ft.
49.5	20	309.845	−.0142	−.0028	+.0221	309.8501	

Mean inclined distance = 309.8510 feet.
Mean horizontal distance = 309.8382 feet.

The value of l used in obtaining the corrections C_t, C_s, and C_p was the mean of the observed distances, and in finding C_g the mean inclined corrected distance was used.

Proceeding in the same manner the corrections were found for divisions II and I, and the sum of the three mean horizontal distances is 922.2235 feet, which is the final horizontal length of the base EG as determined from the observations of October 3, 1888. From similar measurements made on the preceding day its length was found as 922.2255 feet.

During 1889 four measures of this base were made, the results being 922.220 feet, 922.197 feet, 922.221 feet, and 922.217 feet. The mean of these is 922.214 feet, whose probable error is 0.004 feet nearly. The probable relative uncertainty of the mean of these four measures is then about $\frac{1}{230500}$th part of the length. It thus

appears that results of high precision can be obtained with a steel tape whose constants are known. The best time for such work is in still weather when the sky is densely clouded, as then the temperature varies but slightly.

Problem. Compute from the above data and field notes the mean horizontal distances of divisions I and II of the base line EG.

ARTICLE 3. RECONNAISSANCE, AND MARKING OF STATIONS.

Two classes of triangulation are usually recognized in geodetic work: the primary series, which connects directly with the bases and which employs the longest possible lines; and the secondary series, which determines points within the larger triangles. To this is often added a tertiary series which locates stations for the special use of plane table and stadia parties. On the U. S. Coast and Geodetic Survey it is required that in the primary series the probable error of a determined angle should not exceed 0.3 seconds, and that the limit of error in closing a triangle should not exceed 3 seconds, while for the secondary series the limits are 0.7 and 6.0 seconds respectively.

The first field operation is that of reconnaissance, which decides upon the locality of each of the stations. These are to be so selected as to secure the best system of triangles, quadrilaterals, or hexagons, to cover the given area so as to satisfy the prescribed conditions of accuracy and minimum cost, both in the execution of the triangulation and in the subsequent use that is to be made

of it. For the stations of the primary series usually the highest elevations are selected, and accordingly the longest possible lines obtained consistent with the formation of well-proportioned figures. The intervisibility of adjacent stations must of course be insured, and if possible approximate values of the principal angles be determined. For an account of the conditions to be observed in reconnaissance work, see the Coast and Geodetic Survey's *Report* for 1882, pages 151–195, and *Report* for 1885, pages 469–481. See also *Final Results of the Triangulation* of the New York State Survey, Albany, 1887, for an interesting description of the field work of reconnaissance.

The stations are marked by bolts set into the rock, or by stone monuments set in the ground. In the latter case it is customary to bury beneath the monument a bottle or crock whose center marks the center of the station. When this is done the knowledge of the bottle or crock should be concealed from the people of the neighborhood, and it should be covered with a large flat stone having a hole drilled in its upper surface. The top of this flat stone should be two or three feet below the surface of the ground, and upon it the foot of the monument may be set. The centers of the underground mark, of the hole in the flat stone, and of the top of the monument should be in the same vertical. Near the top of the monument "U.S." or other appropriate letters should be cut. Detailed instructions regarding the methods of marking stations may be found in the Reports above quoted. Reference points should be located on surrounding rocks, or by auxiliary monuments, from which bearings and distances are to be measured to the station. Let the geodetic surveyor ever remember that his description of the station should be clear and full, and

that all the marks should be as permanent as it is possible to make them.

The marking of the station in this permanent manner is usually done by the first triangulation party which occupies it, the reconnaissance party merely selecting the place approximately. It is believed, however, that, if the responsibility of marking the station were assigned to those who make the reconnaissance, a better location would often be made.

Problem. Two stations 18 miles apart are 63 and 102 feet, respectively, above the sea-level. The highest point between them is on a ridge 6.5 miles from the first station, its elevation being 73 feet. What should be the heights of the towers at the stations so that the line joining their tops may pass 7 feet above the ridge?

Article 4. Towers and Signals.

A tower is a structure erected over a station for the support of the theodolite and observer. It consists of two parts, an interior tripod to carry the instrument, and an exterior scaffold entirely surrounding the tripod but unconnected with it. At some stations no tower is required, at others it is necessary to build them to great heights in order to see the neighboring stations. In the U. S. Coast and Geodetic Survey's *Report* for 1882, pages 199–208, will be found full directions for constructing towers. Rough towers made of timber cut on the spot can be built for about $1.00 per vertical foot up to heights of 30 feet, exclusive of the cost of the timber. Beyond this height the cost increases more rapidly.

A signal is a pole, target, or other object erected at a station upon which the observer at another station points in measuring the angles. The simplest signal is a pole, but its use involves a liability to error in sighting upon the illuminated side, and hence for the most accurate work plane targets are preferred. These are made of a wooden framework covered with either black or white muslin. For a distance of 15 miles good dimensions for a target are 2 feet in width and 12 feet in height. The target has the disadvantage of requiring to be set anew whenever the observer changes his station, but it has the advantage of being more easily seen than a pole. The old practice of putting a tin cone on a pole in order to render it visible cannot be recommended, except for reconnaissance work.

For long lines neither pole nor target can be recognized, and the heliotrope must be used. This instrument consists essentially of a mirror which reflects the sunlight to the observer's station. The usual size of the mirror is about two inches in diameter, and it should be mounted so that it has a motion about a vertical and a horizontal axis. The mirror may be placed at one end of a board about three feet long upon which are two sights in the same line with the center of the mirror. The sights being pointed at the distant station, the mirror is turned so that the shadow of the rear sight falls upon the front one, and the sunlight then is reflected to the observer, who sees it as a star twinkling in the horizon. As the apparent diameter of the sun is about half a degree, the reflected rays form a cone having the same angle, so that it is only necessary to point the heliotrope within a quarter of a degree of an object in order that the light may reach it. The heliotrope re-

quires, of course, sunlight, but its light pierces the haze and renders observations possible when a pole or target could not be seen.

The longest line of a geodetic triangle thus far observed is one of 192 miles on the Davidson quadrilateral in California. At that distance the heliotrope required a mirror having an area of 77 square inches. For ordinary lines of from 20 to 40 miles in length a combination of both target and heliotrope will be found advantageous, the former being used on cloudy days with clear atmosphere and the latter in sunshine. In this case the heliotroper erects the target over the station, and places his instrument in line in front of it.

Night signals have been successfully used. These are generally large petroleum lamps with reflectors, which are placed in position and lighted by the heliotropers on leaving their stations in the evening. A magnesium tape whose burning is regulated by clockwork has been also employed. Night work should be usually combined with day work, the observer being on duty from noon to midnight.

Problem. Find the diameter of a pole, or the width of a target, which subtends five seconds of arc at a distance of ten miles.

Article 5. Measurement of Angles.

Angles are measured either with a repeating theodolite or with a direction theodolite. Any engineer's transit may be used as a repeating theodolite, and the following notes will have special reference to the work of such instruments. A direction theodolite has no verniers, but the readings are made by several micrometer microscopes placed around the circle.

The method of repeating angles will be familiar, in general, to all who read these pages, but the details may vary slightly with the degree of precision required. It was the practice of the author in the primary triangulation in Pennsylvania to take sets of four repetitions each, the average number of sets for each independent angle being usually about 25. In each set two of the measures were made with the telescope in the direct position, and the other two with it in the reverse position, in order to eliminate any error in the level of its horizontal axis. Both verniers were read in order to eliminate any error of eccentricity in the graduated plate. Usually, also, the sets were taken alternately from left to right and from right to left, in order to balance any errors arising from the use of the clamps or from a motion of the tripod. The different sets were taken upon different parts of the circle, to eliminate any errors in the graduation of the limb. Lastly, the observations were continued over several days, in order that the signals may be seen under different atmospheric conditions. The following is an example of four sets:

Station: Bake Oven. Instrument: Gambey No. 16.

Station observed.	Aug. 25, 1885.	Reps.	Reading.	A	B	Mean	Angle.
	P. M.						
Smith's Gap	3.45		0° 00′	00″	12″	06″.0	
		4					98° 36′ 58″.1
Knob	4.02		394 27	54	63	58 .5	
		4					98 36 55 .5
Smith's Gap	4.06		0 00	12	21	16 .5	
Smith's Gap	4.33		50 29	33	48	40 .5	
		4					98 36 62 .6
Knob	4.37		84 57	45	57	51 .0	
		4					98 36 60 .4
Smith's Gap	4.40		50 29	42	57	49 .5	

When five lines center at a station, as in Figure 14, there are four independent angles only, but it is best to measure all the angles resulting from the combination of these two by two. This gives ten angles to be observed. Bear's Head Station was occupied on July 20, 1885, and the work was completed on July 30, but the weather permitted observations on only eight days, and on five of these measurements could not be made until the afternoon. The total number of measures is seen to be 456, or an average of 114 for each single independent angle. The adjustment of the observed values is made by Least Squares, according to the methods set forth in pages preceding and fully exemplified in Article 7.

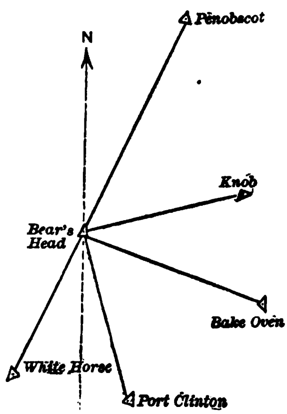

Fig. 14.

Angle at Bear's Head.	No. of Reps.	Observed Value.	Adjusted.
White Horse-Port Clinton	48	41° 15′ 39″.97	40″.33
White Horse-Bake Oven	40	94 47 12 .71	14 .02
White Horse-Knob	48	129 19 47 .57	47 .09
White Horse-Penobscot	40	180 39 48 .43	47 .24
Port Clinton-Bake Oven	48	53 31 34 .27	33 .69
Port Clinton-Knob	48	88 04 05 .75	06 .76
Port Clinton-Penobscot	48	139 24 06 .97	06 .91
Bake Oven-Knob	48	34 32 33 .16	33 .07
Bake Oven-Penobscot	40	85 52 32 .39	33 .22
Knob-Penobscot	48	51 19 59 .71	60 .15

When four stations are to be observed there are six angles to be measured, and for m stations there are

$\frac{1}{2}m(m-1)$ angles. By distributing the work among all of these instead of confining it to the $m-1$ independent angles the field-work is usually expedited, and the final results are more satisfactory, since in the adjusted values the errors of pointing are largely eliminated. When the numbers of repetitions are so nearly equal as in the above instance, the weights of the observed values can fairly be regarded as equal in making the adjustment.

ARTICLE 6. ABSTRACTS OF ANGLES.

In the field note-book the observations are recorded in the order in which they are made, and it is desirable before the occupation of a station is concluded that the results for each angle should be arranged in an abstract and the probable error be computed. Thus the observer gains a clear idea of the precision of each angle and is able to decide whether additional measures are necessary. The weights of the final means are, however, usually assigned from the number of repetitions rather than by the probable errors.

In the permanent triangulation around Lehigh University angle work of a high degree of precision has been done with engineer's transits having verniers reading to minutes or half-minutes. In using such instruments it is easy to estimate closer than the least reading of the vernier, and it has been found possible with 12 sets of four repetitions each to obtain a mean value of an angle whose probable error ranges from one second to three fourths of a second. The following is an example of the abstract of one of these angles, where all the 48 measures were made on the same afternoon:

Station N. Angle P. N. M. L. U. Triangulation.

Oct. 12, 1887.	Reps.	Angle.	v	v^2	Remarks.
2.20 P.M.	4 L. to R.	93° 21′ 11″.25	+ 5.00	25.0	Instrument:
	4 R. to L.	18 .75	− 2.50	6.2	B. & B. Transit,
2.30	4	15 .00	+ 1.25	1.6	No. 715.
	4	22 .50	− 6.25	39.1	Observer:
2.41	4	07 .50	+ 8.75	76.6	C. W. Focht, 2
	4	22 .50	− 6.25	39.1	to 3 P.M.; W. S.
3.00	4	11 .25	+ 5.00	25.0	Davis, 3 to 4 P.M.
	4	22 .50	− 6.25	39.1	Air clear.
3.16	4	11 .25	+ 5.00	25.0	
	4	18 .75	− 2.50	6.2	$n = 12$
3.27	4	18 .75	− 2.50	6.2	$r = 3''.43$
	4	15 .00	+ 1.25	1.6	$r_0 = 0\ .99$
	48	93° 21′ 16″.25 ± 0″.99		290.6 = Σv^2	

Proceeding as in Section IV, page 99, the mean is found, and then the residuals and their squares are formed. The probable error of one observed value is computed to be 3″.43, while that of the final mean is 0″.99, which is a good degree of precision for the instrument used, as its verniers read only to minutes.

A young observer is usually tempted, after having computed the mean and found the probable errors, to reject some of the observations which have the largest residuals, in order thereby to apparently increase the precision of the results. This temptation must be resisted. There are, however, two cases where an observed value may properly be rejected, namely, if it is evidently a mistake, as when the degrees and minutes of the angle are wrong, and if a remark in the note-book shows it was taken under unfavorable conditions. Some observers allow themselves the liberty of rejecting an observation when its residual is greater than five times the computed probable error of a single observation. There

are reasons in favor of this practice (see Merriman's *Method of Least Squares*, page 169), but it is better for the student to avoid it, and to devote his energies to improving his skill in field work.

Article 7. The Station Adjustment.

When the adjusted values of the several angles are of equal weights the method given in Sections VI and VII is to be followed, and it is here only necessary to explain the abridgment whereby the numerical work is simplified. To illustrate, let the data be the same as on page 107, but let x, y, and z be taken to represent the three angles CNM, CNQ, and MNS. Then the five observation equations are

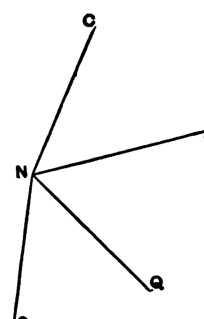

Fig. 15.

$$\begin{aligned} x &= 55^\circ\ 57'\ 58''.68 \\ -x+y &= 48\ \ 49\ \ 13\ .64 \\ y &= 104\ \ 47\ \ 12\ .66 \\ x-y+z &= 54\ \ 38\ \ 15\ .53 \\ z &= 103\ \ 27\ \ 28\ .99 \end{aligned}$$

Now let x_1, y_1, and z_1 be the most probable corrections to be applied to the measured values, so that if x, y, and z be the most probable values,

$$\begin{aligned} x &= 55^\circ\ 57'\ 58''.68 + x_1 \\ y &= 104\ \ 47\ \ 12\ .66 + y_1 \\ z &= 103\ \ 27\ \ 28\ .99 + z_1 \end{aligned}$$

Then substituting these in the observation equations, the latter reduce to

$$
\begin{aligned}
x_1 &= 0''.00 \\
-x_1 + y_1 &= -0\ .34 \\
y_1 &= 0\ .00 \\
x_1 - y_1 + z_1 &= +0\ .52 \\
z_1 &= 0\ .00
\end{aligned}
$$

from which the normal equations are found to be

$$
\begin{aligned}
3x_1 - 2y_1 + z_1 &= +0''.86 \\
-2y_1 + 3y_1 - z_1 &= -0\ .86 \\
x_1 - y_1 + 2z_1 &= +0\ .52
\end{aligned}
$$

the solution of which gives the values

$$x_1 = +0''.15, \qquad y_1 = -0''.15, \qquad z_1 = +0''.11,$$

and therefore the most probable values of x, y, and z are

$$
\begin{aligned}
x &= 55^\circ\ 57'\ 58''.83 = \text{CNM} \\
y &= 104\ \ 47\ \ 11\ .51 = \text{CNQ} \\
z &= 103\ \ 27\ \ 28\ .99 = \text{MNS}
\end{aligned}
$$

and from these, by simple subtraction, the most probable values of the others can be found, namely,

$$
\begin{aligned}
-x + y &= 48^\circ\ 49'\ 13''.68 = \text{MNQ} \\
x - y + z &= 54\ \ 38\ \ 15\ .42 = \text{QNS}
\end{aligned}
$$

The weights of the mean observed values are usually taken as proportional to the number of repetitions. When these are unequal the normal equations are to be

derived by the rule of Section VIII. As an example, let the following be three angles measured at the station O:

$$\begin{aligned} MOA &= 46^\circ\ 53'\ 29''.4 \quad \text{with weight } 4 \\ MOC &= 135\ \ 27\ \ 11\ .1 \quad \text{with weight } 9 \\ AOC &= 88\ \ 33\ \ 41\ .1 \quad \text{with weight } 2 \end{aligned}$$

Now let x and z be the most probable values of any two angles, say of MOA and MOC. Then the observation equations are

$$\begin{aligned} x &= 46^\circ\ 53'\ 29''.4, \quad \text{weight } 4 \\ z &= 135\ \ 27\ \ 11\ .1, \quad \text{weight } 9 \\ z - x &= 88\ \ 33\ \ 41\ .1, \quad \text{weight } 2 \end{aligned}$$

Next let x_1 and z_1 be the most probable corrections to the observed values of x and z, or

$$\begin{aligned} x &= 46^\circ\ 53'\ 29''.4 + x_1, \\ z &= 135\ \ 27\ \ 11\ .1 + z_1, \end{aligned}$$

and let these be substituted in the observation equations, which thus reduce to

$$\begin{aligned} x_1 &= 0''.00, \quad \text{weight } 4 \\ z_1 &= 0\ .00, \quad \text{weight } 9 \\ x_1 - z_1 &= +0\ .60, \quad \text{weight } 2 \end{aligned}$$

From these the normal equations are formed; they are

$$\begin{aligned} 6x_1 - 2z_1 &= +1.20 \\ -2x_1 + 11z_1 &= -1.20 \end{aligned}$$

from which the most probable corrections are

$$x_1 = -0''.06, \qquad z_1 = -0''.77.$$

Finally, the adjusted values of the three angles are

$$\begin{aligned} x &= \text{MOA} = 46^\circ\ 53'\ 29''.34 \\ z &= \text{MOC} = 135\ \ 27\ \ 10\ .33 \\ z - x &= \text{AOC} = 88\ \ 33\ \ 40\ .99 \end{aligned}$$

Problem. Adjust the angles observed at the station Bear's Head, given in Article 5, in two ways: (*a*) taking the weights as equal; (*b*) taking the weights as proportional to the number of repetitions.

Article 8. Excentric Signals and Stations.

Sometimes, owing to obstructions in the line of sight, it is necessary to put a signal a short distance away from the station. When this is done its distance and direction from the station should be measured with care, and then the corrections to be applied to the angles can be computed as soon as the field work has sufficiently progressed so that approximate values of the lengths of the triangle sides can be found. For instance, in 1878, an observer

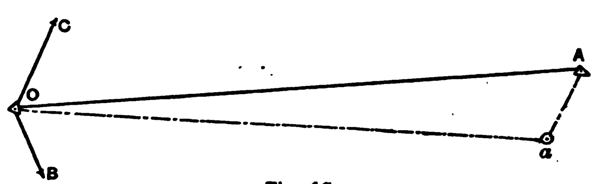

Fig. 16.

at the station O determined the angles BO*a* and CO*a*, where the heliotrope had been placed at *a* instead of at the station A. The distance A*a* was reported as 16 feet 2 inches, and the angle A*a*O as 129° 35′. Later, in 1883, the work had progressed so that OA was found to be 29 556 meters. It is required to compute the angle AO*a*,

so as to determine the corrected values of BOA and COA. By the elementary formulas of trigonometry

$$\sin AOa = \frac{Aa}{AO} \sin AaO,$$

and inserting the data, and making careful use of the special tables for the sines of small angles, there is found $AOa = 26''.63$, which is the value to be added to BOa and to be subtracted from COa.

Sometimes a church spire, or other inaccessible point, is taken for the station, so that the angles cannot be observed except by occupying an excentric station near the true one. In Fig. 17 let A be the station and a the excentric station. The angle MaN having been observed, it is required to find MAN. To do this Aa must be measured and also the angle AaM, between Aa and the left-hand station. Let $Aa = r$, $AaM = \theta$, $AM = m$, $AN = n$, $AMa = M$, and $ANa = N$. Then, as the opposite angles made by the crossing lines are equal, $A + M$ equals $a + N$, and hence the angle MAN is

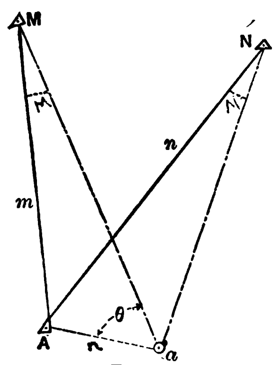

Fig. 17.

$$A = a + N - M,$$

in which N and M are to be computed from

$$\sin N = \frac{r}{n} \sin (a + \theta), \qquad \sin M = \frac{r}{m} \sin \theta,$$

and the signs of N and M will depend upon those of $\sin(a + \theta)$ and $\sin \theta$. As an example (let the student

draw the figure) $a = 93° 56' 25''$, $\theta = 236° 24' 15''$, $r =$ 2.2145 meters, $\log n = 3.9089136$, $\log m = 3.9571308$. Here using the trigonometrical tables for log sines of small angles, there are found $N = -27''.87$ and $M = +52''.82$, whence the required angle A is 93° 56′ 50″.01.

Excentric signals and stations should be avoided. Indeed it is best that heliotropers should not know that their instruments can be set anywhere except directly over the station, otherwise they will often be tempted to set them excentrically, and may not be able to take the necessary measurements to correct the angles. Excentric stations are now and then required, and here there is less chance for error.

Problem. In Figure 16 let $OA = D$, $Aa = d$, and $OAa = \beta$. Deduce the formula for finding the angle $AOa = \alpha$. Compute α when $D = 64\,382$ feet, $d = 3.45$ feet, and $\beta = 104° 20'$. Also for the same distances when $\beta = 104° 30'$.

Article 9. The Triangle Adjustment.

For most computations made in the field the spherical excess need not be regarded, but in the final computations at the close of the season it should be found for each triangle. The spherical excess is approximately proportional to the area of the triangle at the rate of about one second of angle for each 75½ square miles of area. The spherical excess, in seconds, of a triangle whose sides in meters are a and b, with the included angle C, is computed from the formula

$$\epsilon = m \,.\, ab \sin C,$$

in which m depends upon the shape and size of the

spheroid and upon the mean latitude of the triangle. For CLARKE's spheroid the following values of m are given in U. S. Coast and Geodetic Survey's *Report* for 1882, p. 170.

Latitude.	log m.	Latitude.	log m.
31° 00′	1.40537	40° 00′	1.40451
30	533	30	446
32 00	528	41 00	441
30	524	30	436
33 00	519	42 00	431
30	514	30	426
34 00	509	43 00	420
30	505	30	415
35 00	500	44 00	410
30	495	30	405
36 00	491	45 00	400
30	486	30	395
37 00	481	46 00	390
30	476	30	385
38 00	471	47 00	380
30	466	30	375
39 00	461	48 00	369
30	1.40456	30	1.40364

In order to find ϵ, the values of a and b are first to be approximately determined by Article 10, and then the approximate latitudes of the three stations by Article 11. Five-place logarithmic tables will be sufficiently precise for these preliminary computations, and the latitudes need be found only to the nearest minute.

As an example let the data for a triangle be:

Stations.	Angles adjusted at Stations.	Approximate Distances.	Approximate Latitudes.
Pimple Hill	49° 04′ 50″.13	27 540 meters	41° 02′
Smith's Gap	90 21 25 .53	36 440 "	40 49
Bake Oven	40 33 46 .91	23 700 "	40 45
	Sum = 180° 00′ 02″.57		Mean L. = 40° 52′

Now C can be taken as any one of these angles and a and b as the two adjacent sides. It is advisable to make two check computations for ϵ, thus:

log m	1.40441	log m	1.40441
$a = 36\,440$	4.56158	$a = 23\,700$	4.37475
$b = 23\,700$	4.37475	$b = 27\,540$	4.43996
$C = 49° \; 04' \; 50''$	9.87831	$C = 29° \; 21' \; 30''$	9.99999
$\epsilon = 1''.66$	0.21905	$\epsilon = 1''.66$	0.21911

The adjustment of the angles of a triangle usually consists in applying one third of the error to each of the given angles to obtain the spherical angles, and then subtracting one third of the spherical excess to obtain the plane angles. For instance, using the above triangle:

Stations.	Angles adjusted at Stations.	Spherical Angles.	Plane Angles.
Pimple Hill	49° 04′ 50″.13	50″.83	50″.28
Smith's Gap	90 21 25 .53	25 .22	24 .66
Bake Oven	40 33 46 .91	46 .61	46 .06
Sum =	180° 00′ 02″.57	01″.66	00″.00
$180° + \epsilon$ =	180 00 01 .66		
Error =	+ 0″.91		

When the weights of the angles adjusted at the stations are very unequal, it may be advisable to proceed as in Section VIII, page 112, to find the spherical angles. Thus if the triangle KPS have 8 sets measured on K and on both P and S, the adjustment is:

Stations.	Weights.	Angles adjusted at Stations.	Spherical Angles.	Plane Angles.
K	1	41° 20′ 34″.34	35″.52	34″.91
P	6	79 03 41 .73	41 .93	41 .32
S	6	59 35 44 .18	44 .38	43 .77
	Sum =	180° 00′ 00″.25	01″.83	00″.00
	$180° + \epsilon$ =	180 00 01 .83		
	Error =	− 01″.58		

Problem. Compute the spherical excess for the above triangle KPS, taking the mean latitude as 40° 55′ and PS = 23 700 meters. Adjust the triangle (*a*) when the three angles have equal weights; (*b*) when the weights of K, P, and S are 1, 3, and 4, respectively.

Article 10. Computation of Triangle Sides.

The computation of the sides of a triangle is a simple matter, one side being first known from preceding computations or having been measured as a base line. The following is a convenient form for arranging the work:

Stations.	Plane Angles.	Logarithms.	Distances.
	S to B	4.4398898	27535.3
P	49° 04′ 50″.28	0.1216896	
S	90 21 24 .66	9.9999916	
B	40 33 46 .06	9.8131011	
	P to S	4.3746805	23696.3
	P to B	4.5615710	36439.4

Here the stations are arranged in azimuthal order, that being placed first which is opposite to the given side, the length of this and its logarithm being put on the top

line. Opposite the second and third angles are written their logarithmic sines, and opposite the first angle the arithmetical complement of its logarithmic sine. Now, to find the log of PS the logarithm opposite S is to be covered with a lead-pencil and the other three logarithms added. So to find the log of PB the logarithm opposite B is to be covered and the other three logarithms added. Lastly, the distances corresponding to these logarithms are taken from the table.

If the precision of angle work really extends to hundredths of seconds, as it does on primary triangulation, a seven-place table of logarithms will be needed. Six-place tables are rarely found conveniently arranged for rapid and accurate computation. For a large class of work five-place tables are sufficiently precise.

The unit of distance employed by the Coast and Geodetic Survey is the meter. The following constants are adopted, from the latest comparisons, to convert meters into feet and miles when necessary:

$$\text{Meters} \times 3.280869 = \text{Feet.}$$
$$\text{Meters} \times 0.000621377 = \text{Miles.}$$

If the logarithm of the distance in meters be known, the conversion may be made as follows:

$$\text{Log meters} + 0.5159889 = \text{Log feet.}$$
$$\text{Log meters} + 6.7933550 = \text{Log miles.}$$

Problem. Compute the lengths of KS and KP, using the data of the last Article and taking the length of PS as 23696.3 meters.

ARTICLE 11. THE L M Z PROBLEM.

At a given station let L be the latitude, M the longitude, and Z the azimuth of a line to a second station. At the second station let L′ be the latitude, M′ the longitude, and Z′ the azimuth of a line to the first station. Let k be the distance between the two stations. Let L, M, and Z be given; it is required to compute L′, M′, and Z′.

Let dL, dM, and dZ be the differences of latitude, longitude, and azimuth between the two stations. When k does not exceed about 12 miles, or 20 000 meters, these differences may be computed by the following formulas, whose demonstration is given in U. S. Coast and Geodetic Survey's *Report* for 1884, pages 323–375, where also extended tables of the factors B, C, etc., may be found:

$$-d\mathrm{L} = k \cos \mathrm{Z} \,.\, \mathrm{B} + k^2 \sin^2 \mathrm{Z} \,.\, \mathrm{C} + h^2 \,.\, \mathrm{D};$$

$$+d\mathrm{M} = \frac{k \sin \mathrm{Z} \,.\, \mathrm{A}'}{\cos \mathrm{L}'};$$

$$-d\mathrm{Z} = d\mathrm{M} \,.\, \sin \tfrac{1}{2}(\mathrm{L} + \mathrm{L}').$$

In these formulas h represents the term k cos Z . B. The factors B, C, and D are to be taken from the tables for the argument L, and A′ is to be taken for the argument L′. The signs of cos Z and sin Z are to be carefully used to determine the final signs of the required quantities. The table of factors is arranged for the meter as a unit of length, and hence k must always be

expressed in meters. These formulas may also be used for secondary triangulation when k is greater than 12 miles.

TABLE OF L M Z FACTORS.

Latitude.	10 + log A′.	10 + log B.	10 + log C.	10 + log D.
31°	8.509 3363	8.511 5054	1.18416	2.3382
32	3134	4368	.20108	.3460
33	2901	3669	.21772	.3532
34	2665	2959	.23409	.3597
35	2425	2239	.25024	.3656
36	2182	1510	.26617	.3709
37	1936	0772	.28193	.3756
38	1687	0027	.29753	.3797
39	1437	8.510 9275	.31299	.3833
40	1184	8517	.32833	.3863
41	0930	7755	.34358	.3888
42	0675	6989	.35875	.3907
43	0419	6220	.37386	.3921
44	0162	5449	.38894	.3930
45	8.508 9904	4677	.40400	.3933
46	9647	3905	.41906	.3932
47	9390	3134	.43414	.3924
48	9133	2364	.44926	.3912
49	8878	1598	.46443	.3894
50	8623	8.510 0835	1.47968	2.3871

In using the above formulas dL is first computed, and L' is known by

$$L' = L + dL.$$

Then, with L' as an argument, A' is taken from the table and dM is computed, whence

$$M' = M + dM.$$

Lastly, using dM, the third formula gives dZ, and then

$$Z' = Z + 180^\circ + dZ.$$

As an example, let the latitude and longitude of the station A be given and the azimuth and distance from A to R, and let it be required to find the L′, M′, and Z′ for the station R. The given values of L, M, and Z are first arranged as follows, with blank spaces where the results of the computations are to be recorded:

Z	Station A to Station R	193° 56′ 28″.11
180° + dZ		180
Z′	Station R to Station A	

L	40° 36′ 22″.452	Station A	M	75° 22′ 51″.150
dL		$k = 1726.598$	dM	
L′		Station R	M′	

The computations for the example in hand, and the whole form completely filled out, are shown below. It is best to take B, C, and D from the table of factors at the same time, as also cos Z and sin Z. In this case cos Z is negative and thus h and $-dL$ are negative, whence dL is positive. After having found L′ the computation of dM is made, and its sign is negative, because sin Z is negative, whence dM is also negative. Lastly, $-dZ$ is found to be negative, which gives dZ as positive. The signs of these quantities must always be carefully regarded.

A check computation for L′ and M′ should always be made by taking a line from another station where L, M, and Z are also known. The values of Z′ found in the two computations can be checked by taking their differ-

ence, which should equal the spherical angle between the two lines.

Z	Station A to Station R	193° 56′ 28″.11
180° + dZ		+180 +11 .52
Z′	Station R to Station A	13° 56′ 39″.63

L	40° 36′ 22″.452	Station A	M	75° 22′ 51″.150
dL	+ 54 .327	k = 1726.598	dM	− 17 .698
L′	40° 37′ 16″.779	Station R	M′	75° 22′ 33″.452

k	3.2371912	k^2	6.47438		
cos Z	9.9870151	sin² Z	8.76376	h^2	3.4700
B	8.5108056	C	1.33758	D	2.3879
h	1.7350119		6.57572		5.8579
1st term	− 54″.327	2d term	+ 0″.0004	3d term	+ 0″.0000
2d and 3d terms	0 .000	k	3.2371912		
− dL	− 54″.327	sin Z	9.3818819		
		A′	8.5091027	dM	1.247917
½dL	27″.163	cos L′ a.c.	0.1197416	sin ½(L + L′)	9.813528
½(L + L′)	40° 36′ 49″.615		1.2479174		1.061445
		+ dM	− 17″.698	− dZ	− 11″.52

Problem. Given for Station L the latitude = 40° 36′ 31″.935, the longitude = 75° 22′ 07″.482, the distance to R = 1511.954 meters, and the azimuth to R = 156° 11′ 15″.50; to compute L′, M′, and Z′ for the station R.

Problem. Make the two L M Z computations for the station R in the triangle BSR, the data being:

Station.	L.	M.
B	40° 44′ 54″.109	75° 44′ 02″.222
S	40 49 21 .787	75 25 21 .906

Line.	Z.	k.
BR	294° 52′ 30″.39	33389.85 meters
SR	349 57 34 .13	22710.6 "

ARTICLE 12. PRECISE PLANE TRIANGULATION.

A short base, say about 1000 feet in length, measured in the manner explained in Article 2, may serve as the foundation for a precise triangulation if the angles be observed with the care indicated in Article 5. The stations should be permanently marked by massive stone monuments or by holes in rock. The signals should be iron rods, as small as consistent with visibility, held truly vertical. The field work is then like that of a geodetic triangulation, but on a diminutive scale. In order that the triangulation may be properly oriented in the computations with respect to the meridian, it is desirable that the azimuth of one of the sides should be determined by astronomical observations. The azimuths of the other sides may then be deduced by the rule $Z' = 180° + Z$, thus neglecting the convergence of the meridians, or, in other words, regarding the earth as a plane. The coördinates of the stations, expressed usually in feet, will be measured parallel and normal to the meridian from some assumed origin.

Such a system, for example, is shown in Fig. 18, which represents the permanent triangulation around Lehigh University. The length of the base line EG, as determined from a number of measures, is 922.219 feet. The stations P and R were spires which could not be occupied. At each of the other stations the observed angles were adjusted as in Article 5, and on completing the work it was found that the error of closure in the triangle HKL was $0''.8$, and the sum of the interior angles of the polygon AEGHKMNQA was $8''.1$ less than the theoretical sum 1080°. Another check was obtained by

measuring the line HK with the same precision as the base EG, and comparing it with the value as computed through the triangles; this gave a discrepancy of 0.028 feet when using only the three triangles lying between

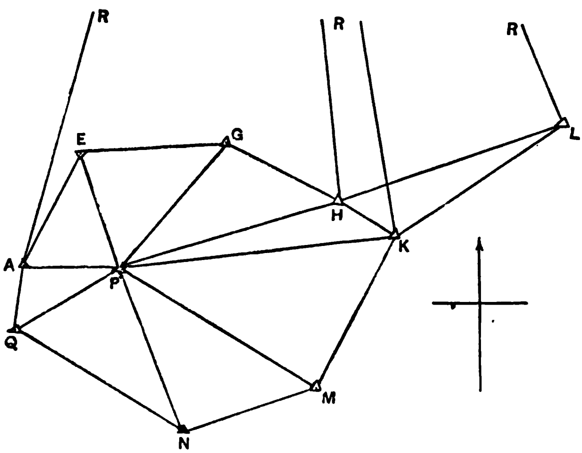

Fig. 18.

them, and 0.033 feet when passing around the polygon the other way. These discrepancies were afterwards removed by adjusting the whole figure by the Method of Least Squares.*

At the station K the azimuth of the line KR was observed astronomically, and from this and the adjusted angles the plane azimuths were found for the other lines. These plane azimuths will be called directions in order that no confusion may result. Afterwards the geodetic azimuths were computed by Article 11, and the comparison of these with the directions for a few

* See the paper on precise triangulation by H. S. JACOBY in the *Lehigh Quarterly*, January 1891.

lines will be interesting as showing how the curvature of the earth can be detected on lines so short.

Lines.	Distances (feet).	Directions.	Azimuths.
A to R	5664.740	193° 56′ 47.″7	193° 56′ 28″.1
A to P	593.552	271 26 04 .5	271 25 44 .9
H to R	5090.855	173 08 48 .8	173 08 45 .8
H to K	410:019	300 23 44 .3	300 23 41 .3
K to H	410.019	120 23 44 .3	120 23 44 .3
K to R	5348.995	169 38 55 .3	169 38 55 .3
R to K	5348.995	349 38 55 .3	349 38 47 .2
L to K	1268.582	55 12 39 .6	55 12 48 .4
L to R	4960.523	156 11 06 .7	156 11 15 .5

The following are plane coördinates of a few of these stations, which were computed from the triangle sides and directions, the coördinates of K being first assumed:

Stations.	Latitudes (feet)	Longitudes (feet).
A	764.068	4326.412
H	1207.438	2353.673
K	1000.000	2000.000
L	1723.851	958.203
R	6261.888	2961.398

By the help of the above data absolute checks are obtained on the accuracy of all topographic work which connects with the triangulation. For instance, if a transit is set at A, the plane azimuth, or direction, is found at once by pointing either on R or P; then running a traverse to another station, say K, the field-work is checked by the agreement of the observed direction KR with that which is known. Again, in the computation of the traverse, the computed difference of latitude and longitude between A and K are also checked. Lastly, by whatever method the traverse be plotted, it falls into its proper

position on the projection where the stations are first located, and errors in plotting are easily detected.

If such a triangulation can be connected with one of the stations of a geodetic system, the geographical positions of its stations can be computed by the methods of Article 11. This has been done for the case of Figure 18, as the spire R is located by the work of the U. S. Coast and Geodetic Survey. The following are the results for a few of the stations:

Stations	Latitudes.	Longitudes.
A	40° 36′ 22″.452	75° 22′ 51″.150
H	40 36 26 .833	75 22 25 .574
K	40 36 24 .783	75 22 20 .989
L	40 36 31 .935	75 22 07 .482
R	40 37 16 .778	75 22 33 .452

Problem. Transform the difference of geodetic latitude between A and H into feet by the help of the table at the foot of page 59, and compare the result with the difference as deduced from the plane coördinates.

Article 13. Spherical Rectangular Coördinates.

The system of geographical coördinates is not generally convenient for the use of local surveyors. The system of plane coördinates, illustrated in the last Article, is satisfactory for a small area, but when extended over a large territory discrepancies arise because of the erroneous assumption involved in the word "plane." The method of spherical rectangular coördinates, now to be explained, is a compromise between the two systems, the surface of the earth being regarded as a sphere for the area involved, the latitudes and longitudes being

measured in linear units from an assumed origin, and the azimuths being really directions with reference to a principal meridian. This method has long been well known in Europe, but it was only lately introduced into this country, the first publication being by HORACE ANDREWS, C.E., in Appendix F of the *Final Results of the New York State Survey*, 1887. It can be applied to a territory of several thousand square miles, and will furnish coördinates which may at any place be used by a local surveyor exactly like plane coördinates.

Let x and y be the spherical rectangular coördinates of a station measured from an origin near the middle of the area covered by the system; x is reckoned positive to the south and negative to the north; y is reckoned positive to the west and negative to the east. Let α be the direction-angle at the station of any line to a second station, namely, the angle which the line makes with an arc drawn through the station parallel to the meridian through the origin. Let k be the distance to a second station, and let $n = k \sin \alpha$ and $m = k \cos \alpha$. Then x' and y', the spherical rectangular coördinates of the second station, are given by the formulas

$$y' = y + n - \frac{m^2 y}{2r^2} - \frac{m^2 n}{6r^2},$$

$$x' = x + m + \frac{m y'^2}{2r^2} - \frac{m n^2}{6r^2}.$$

Let α' be the direction-angle at the second station of a line drawn to the first station; it is found by the formula

$$\alpha' = \alpha + 180° - \frac{my}{r^2 \sin 1''} - \frac{mn}{2r^2 \sin 1''}.$$

In these formulas r is the mean radius of the earth for the territory covered by the system, or the radius of the osculatory sphere to the CLARKE spheroid for the assumed origin. Its value in meters may be computed for any given latitude from the expression

$$\log r = 15.3144251 - \tfrac{1}{2}(a' + b),$$

in which a' and b are the values of $10 + \log A'$ and $10 + \log B$ as given in the table of LMZ factors in Article 11. The reciprocals of $2r^2$ and $6r^2$ are small quantities, which are easily found with five-place logarithms; for latitude 40° the reciprocal of $2r^2$, in meters, is 0.000 000 000 000 012 305.

The above formulas for y' and x' are seen to be the same as for plane rectangular coördinates with the addition of terms which take account of the curvature of the sphere. For a triangulation with sides as short as in the last Article these terms will be inappreciable, but for lines five miles in length they become sensible. Four-place logarithms will usually be sufficiently precise in computing these terms, and the work can be carried on with much greater rapidity than the L M Z problem.

Problem. Let DE be a line in about latitude 39° 55′, its length being 16670.5 meters, and the direction-angle at D being 31° 59′ 48″.83. Taking D as the origin of coördinates, compute the spherical rectangular coördinates of E and the back direction-angle at E. Check the work by taking the square root of the sum of the squares of the computed coördinates.

ARTICLE 14. THE THREE-POINT PROBLEM.

This problem is frequently employed to locate the positions of subordinate stations after the regular triangulation work has been completed. In Fig. 19 let A, B, and C be the three stations whose positions are known, so that the angle ACB is known, as also the distances AC and BC. At P the two angles P_1 and P_2 are measured, and it is required to compute the distances PA, PB, and PC. To do this let x be the angle PAC, and y the angle PBC; then, as PC is common to the two triangles,

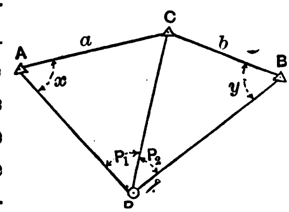

Fig. 19.

$$\frac{a \sin x}{\sin P_1} = \frac{b \sin y}{\sin P_2};$$

also $$x + y = 360 - C - P - P' = 2S.$$

From these two equations the following solution is deduced: first, compute Z from

$$\tan Z = \frac{a \sin P_1}{b \sin P_2};$$

secondly, find T from

$$\tan T = \cot (Z + 45^\circ) \tan S,$$

and then the angles x and y are

$$x = S + T, \qquad y = S - T,$$

from which, with the other data, the sides PA, PB, and

PC are easily computed. As PC is common to both triangles, its value can be found in two ways, thus verifying the work.

As an example let the data be $a = 6273.8$ feet, $b = 7289.0$ feet, $C = 111^\circ\ 10'\ 54''$, $P_1 = 50^\circ\ 06'\ 12''$; $P_2 = 43^\circ\ 50'\ 38''$. From these $S = 77^\circ\ 26'\ 08''$. Using seven-place logarithms the value of Z is found to be $43^\circ\ 37'\ 49''.6$, and then that of T to be $6^\circ\ 07'\ 21''.7$, whence

$$x = 83^\circ\ 33'\ 29''.7, \qquad y = 71^\circ\ 18'\ 46''.3.$$

Lastly, from the triangle PAC,

$$PA = 7194.87 \text{ feet}, \qquad PC = 8999.89 \text{ feet},$$

and from the triangle PBC,

$$PB = 8107.98 \text{ feet}, \qquad PC = 8999.89 \text{ feet},$$

and the coördinates of P can be computed at once when those of A, B, or C are given.

Problem. Given AC = 3136.9 feet, BC = 3644.5 feet, $C = 111^\circ\ 10'\ 54''$, $P_1 = 104^\circ\ 00'\ 00''$, and $P_2 = 100^\circ\ 20'\ 00''$. Draw the figure, and compute PA, PB, and PC.

Article 15. Miscellaneous Exercises.

(1) In Figure 13 it is required to compute the azimuth at Topton of the line Topton–Big Rock, the positions of the two stations being as follows:

Station.	Latitude.	Longitude.
Big Rock	40° 33′ 53″.732	75° 26′ 16″.422
Topton	40 28 37 .491	75 40 58 .251

For the answer see *Annual Report* of Geological Survey of Pennsylvania for 1885, pages 681–707.

(2) Demonstrate the formulas for the corrections for temperature, sag, pull, and grade in Article 2.

(3) In a quadrilateral GRPW there are known the angles RGW $= 53° 31'$, GWR $= 56° 42'$, RWP $= 41° 20'$, PRW $= 45° 53'$, WRG $= 69° 47'$. It is required to compute the angles RGP and WGP to the nearest minute.

(4) What angle is subtended by a tower fifteen feet in diameter at a distance of forty miles?

(5) State observation equations for adjusting the angle measurements at Bear's Head station.

(6) Forty sets give a mean with a probable error of one second. How many additional sets are needed to reduce the probable error to one half a second?

(7) Make the station adjustment for Figure 15, taking x, y, and z to represent the angles CNM, MNS, and QNS.

(8) In Figure 17 let $a\mathrm{M} = m'$ and $a\mathrm{N} = n'$. Let r, θ, m', n', and a be known. Prove that $\mathrm{A} = a + \mathrm{N} - \mathrm{M}$, in which N and M are to be found from

$$\tan \mathrm{N} = \frac{r \sin (a + \theta)}{n' - r \cos (a + \theta)}, \qquad \tan \mathrm{M} = \frac{r \sin \theta}{m' - r \cos \theta}.$$

(9) In the triangle BSR the station R is a spire where angles cannot be observed. The angles adjusted at the stations B and S are

$$\mathrm{B} = 42° 25' 34''.97, \qquad \mathrm{S} = 82° 41' 33''.11,$$

and the spherical excess is $1''.57$. Find the plane and spherical angles of the triangle.

(10) Compute the distances BR and SR of the above triangle BSR, the distance BS being 27535.3 meters.

(11) Make the two L M Z computations for the last problem of Article 11 by the method for primary triangulation given in U. S. Coast and Geodetic Survey's *Report* for 1884, using the form on page 328.

(12) Compute the distance between the stations A and K from their linear coördinates as given in Article 12. Compute the distance from their geographical cöordinates, using the method given in the U. S. Coast and Geodetic Survey's *Report* for 1884, page 330.

(13) For a full exposition of the method of spherical rectangular coördinates see JORDAN'S *Handbuch der Vermessungskunde*, Volume II, Chapter VI.

(14) The n-point problem arises when $n-1$ angles are measured at a point P between n stations whose positions are known. For the solution by the Method of Least Squares see JORDAN's book, quoted above, and also articles in that excellent journal *Zeitschrift für Vermessungswesen.*